J. DIFFERENTIAL GEOMETRY
40 (1994) 505-562

COMPUTING SPECTRAL FLOW VIA CUP PRODUCTS

PAUL KIRK & ERIC KLASSEN

1. Introduction

In this paper we investigate the spectral flow of the "Dirac" operator $D_A = *d_A - d_A*$ acting on $0+2$ forms on a 3-manifold M as A varies in the connections in an $SU(2)$-bundle P over M. This operator arises as the tangential operator (in the sense of [1]) for the signature and self-duality operators on a 4-manifold, and as a stabilization of the Hessian of the Chern-Simons function on the space of gauge equivalence classes of connections on P. In the case where A is flat, it is a square root of the (twisted) Laplacian Δ_A acting on even forms.

Our main contribution to the calculation of the spectral flow of these operators is as follows. If A_t is a path of flat connections on a manifold M which either has boundary (Theorem 4.8) or is closed (Theorem 5.1), then we show that computation of the "local contribution to spectral flow at time $t=0$" (i.e., the slopes of the eigenvalues that are crossing 0 at $t=0$) can be reduced to a cup product computation in the cohomology of M with twisted coefficients. Later in the paper, we carry out these computations using group cohomology and use the results to compute spectral flow for various arcs of connections of torus bundles over the circle S^1.

The spectral flow of D_A as A varies along a path of connections arises in several ways in topology: If M is a homology 3-sphere, it gives the grading of a flat connection on M, viewed as a generator for the chain complex defining Floer homology. It gives the dimension (mod 8) of the moduli space of those self-dual connections on $M \times \mathbb{R}$ with certain prescribed limiting values. It also enters in the stationary phase "approximation" of Witten's 3-manifold invariants; the stationary phase formula states that for large k Witten's invariant $Z_k = \int_{\mathscr{B}} e^{k\, i\, cs}$ is dominated by the sum over the flat connections of terms involving various invariants of the flat connection, in particular the spectral flow to the trivial connection.

Our aim is to give a description of the spectral flow between two flat connections on M when M is cut along a surface $\Sigma \subset M$ in terms

Received September 29, 1992 and, in revised form, November 29, 1993. Both authors supported by NSF Mathematical Sciences Postdoctoral Fellowships.