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HARMONIC MAPS, LENGTH, AND ENERGY IN TEICHMÜLLER SPACE

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Abstract

The limiting behavior of high-energy harmonic maps between closed hyperbolic surfaces is analyzed. In general a measured foliation on the domain is shown to be mapped very nearly (exponentially in the energy) to its geodesic representative in the range. This foliation is in fact the horizontal foliation Φ_h of the Hopf differential Φ of the harmonic map. Φ_h is also characterized as nearly maximizing (up to an additive constant) the ratio of squared hyperbolic length in the range to extremal length in the domain, among all simple closed curves in the domain. The same ratio gives the energy of the map up to an additive constant. This can be viewed as an analogy to other canonical maps between surfaces, for which different optimization problems are characterized by corresponding length-ratio maximizations.

In addition, the asymptotics of a family of harmonic maps obtained when the domain surface is varied along a classical Teichmüller ray are studied. As expected, the limiting Hopf foliation and the foliation determining the ray are equivalent as topological (not measured) foliations.

1. Introduction

The Teichmüller space of a surface can be viewed as the space of complex (or conformal) structures on the surface, or alternatively as the space of hyperbolic structures on it. A key feature in the study of Teichmüller spaces is the construction of comparison maps between surfaces which are "optimal" in some sense appropriate to the point of view. This paper studies *harmonic maps*, which are obtained by minimizing *energy* over a homotopy class of maps between surfaces. The energy of a map between Riemannian manifolds is the integral over the domain of the squared derivative of the map (see §3 for a precise definition). In particular, we analyze the approximate behavior of such maps (in the case of closed surfaces) when their energy is high or, equivalently, when the domain and range are far apart as points in Teichmüller space. We obtain, in Theorems 7.1 and 7.2, analogies between the behavior of harmonic maps and

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