J. DIFFERENTIAL GEOMETRY
26 (1987) 367–370

AN EXAMPLE OF A COMPACT CALIBRATED MANIFOLD ASSOCIATED WITH THE EXCEPTIONAL LIE GROUP G_2

MARISA FERNÁNDEZ

1. Introduction

Recently Harvey and Lawson have introduced the concept of calibrated geometries [10], [11]. In [11] they study manifolds which have a distinguished closed differential form, which allows them to generalize Wirtinger's inequality [15]. The primary example of a calibrated geometry is a Kähler manifold, the distinguished form being the Kähler form. However there are other interesting calibrated geometries, for example those defined in [8] and [10]. To say that a 7-dimensional manifold has holonomy group a subgroup of G_2 is the same thing as saying that it has a 2-fold parallel vector cross product; that is, there exists a 2-fold vector cross product such that the associated 3-form ϕ is both closed and coclosed. (If ϕ is closed the manifold is called an "associative 7-manifold" while if ϕ is coclosed the manifold is called a "coassociative 7-manifold" in the terminology of [10], [11].)

Of particular interest are compact calibrated geometries. In addition to compact Kähler manifolds, there is the compact symplectic manifold of Kodaira-Thurston [12], [14]. In the present note we shall give an example of a compact 7-dimensional manifold V^7 with a 2-fold vector cross product such that the associated 3-form is closed; thus V^7 satisfies a natural weakening from "Hol $\subset G_2$" to "associative." This manifold is a G_2 analog of a symplectic manifold, and like Kodaira-Thurston's manifold it can be realized as the quotient of a nilpotent Lie group by a discrete subgroup.

The Kodaira-Thurston manifold T cannot be Kählerian for topological reasons. In fact there are two ways to prove that T cannot be Kählerian. The easiest way is to observe that the first Betti number of T is 3, whereas any compact Kähler manifold has even first Betti number. A second method

Received May 1, 1985 and, in revised form, March 4, 1986.