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A REMARK ON THE SYZYGIES OF THE GENERIC CANONICAL CURVES

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Let C be a genus g nonhyperelliptic curve. Consider the canonical ring

$$R = \bigoplus_n H^0(\omega_C^n).$$

Set $V = H^0(\omega_C)$ and let S be the polynomial ring Symm(V). Then R can be regarded as a graded S-module. Let $\mathbf{C} = S/\mu$, where μ is the irrelevant ideal of S. Then $\mathbf{C}$ has a minimal graded Koszul resolution:

$$0 \to \wedge^g V \otimes S(-g) \to \cdots \to V \otimes S(-1) \to S \to \mathbf{C} \to 0.$$

$K_{p,q}(C)$ is defined to be the Koszul cohomology group $K_{p,q}(R)$ [1, §1] which is isomorphic to the homogeneous degree $p+q$ part of $\mathrm{Tor}_p^S(R, \mathbf{C})$. Observe that if

$$0 \to L_{g-2} \to \cdots L_1 \to L_0 \to R \to 0$$

is a minimal graded free resolution of R, then $L_p \otimes \mathbf{C} \simeq \mathrm{Tor}_p(R, \mathbf{C})$.

Mark Green conjectures that if C is generic, then $K_{p,2}(C) = 0$ for $p \leqslant [(g-3)/2]$, [1, 5.6]. It is elementary to show that $K_{p,j}(C) = 0$ for $j \geqslant 3$ (Proposition 2). Now one observes that $K_{1,2}(C) = 0$ is equivalent to Petri's theorem, which says that the homogeneous ideal of C in $\mathbb{P}(V)$ is generated by quadrics. In [2], Green and Lazarsfeld showed that if the Clifford index of C is less than or equal to m, then $K_{m,2}(C) \neq 0$. Green conjectures that the converse is also true [1, 5.1].

In this paper, we study the Koszul cohomologies of a generic curve by the degeneration method. We show that if $K_{p,2}(X) = 0$ for a curve of genus n, then $K_{p,2}(C) = 0$ for a generic curve of genus m, if $m \equiv n \pmod{p+1}$ and $m \geqslant n$.

With the aid of the computer program Macaulay, Bayer, and Stillman had showed that if C is generic and $g \leqslant 12$, then $K_{p,2}(C) = 0$ for $p \leqslant [(g-3)/2]$. Using their results, we prove that $K_{2,2}(C) = 0$ for $g \geqslant 7$ and $K_{3,2}(C) = 0$ for

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