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COLLAPSING RIEMANNIAN MANIFOLDS WHILE KEEPING THEIR CURVATURE BOUNDED. I.

JEFF CHEEGER & MIKHAEL GROMOV

0. Introduction

Let Y^n be a complete connected riemannian manifold, and $p \in Y^n$. The injectivity radius, i_p, of the exponential map at p is defined to be the smallest r such that $\exp_p|\overline{B_r(p)}$ fails to be a diffeomorphism onto its image. The present paper is the first of two which are concerned with the situation in which the size of injectivity radius is "small" relative to the curvature.

In this part I, we show that if a smooth manifold X^n admits a certain topological structure called an *F-structure of positive rank*, then X^n also admits a family of metrics, g_δ, such that as $\delta \to 0$, i_p *converges uniformly to zero at all points*, p, but the curvature, K_α, stays bounded (independent of p and δ). Such a family of metrics is said to *collapse* with bounded curvature (by rescaling, one can assume $|K_\delta| \leqslant 1$).

In part II we prove a sort of strengthened converse to this collapsing result. A riemannian manifold Y^n is said to be *ε-collapsed* if $i_p < \varepsilon$ for all p. Intuitively, such a manifold appears to have dimension $< n$ if one examines it on a scale $\gg \varepsilon$. We show that in each dimension, there exists a *critical radius*, $\varepsilon(n)$, such that if Y^n is $\varepsilon(n)$-collapsed and $|K| \leqslant 1$, then Y^n admits an F-structure of positive rank. Thus, if Y^n admits a metric which is sufficiently collapsed, it actually admits a family of metrics which collapse with bounded curvature.

An F-structure on a space, X, is a natural generalization of a torus action. Different tori (possibly not all of the same dimension) act locally on finite covering spaces of subsets of X. These local actions satisfy a compatibility condition, which insures that X is partitioned into disjoint "orbits." The F-structure is said to have *positive rank* if all orbits are of positive dimension.

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