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SUPERSYMMETRY AND MORSE THEORY

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Abstract

It is shown that the Morse inequalities can be obtained by consideration of a certain supersymmetric quantum mechanics Hamiltonian. Some of the implications of modern ideas in mathematics for supersymmetric theories are discussed.

1. Introduction

Supersymmetry is a relatively recent development in theoretical physics which has attracted considerable interest and has been actively developed in several different directions [17], [18].

A number of concepts in modern mathematics have significant applications to supersymmetric quantum field theory [22]. Conversely, as we will see in this paper, supersymmetry has some interesting applications in mathematics. The purpose of this paper is to describe some of those applications and to make the notions of "supersymmetric quantum mechanics" and "supersymmetric quantum field theory" accessible to a mathematical audience.

The mathematical applications in §§2 and 3 will be self-contained. However, it may be useful to first make a few remarks about some of the relevant aspects of supersymmetry.

In any quantum field theory, the Hilbert space $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, where $\mathcal{H}^+$ and $\mathcal{H}^-$ are the spaces of "bosonic" and "fermionic" states respectively. A supersymmetry theory is by definition a theory in which there are (Hermitian) symmetry operators Q_i, $i = 1, \cdots, N$, which map $\mathcal{H}^+$ into $\mathcal{H}^-$ and vice-versa.

Let us define the operator $(-1)^F$ which distinguishes $\mathcal{H}^+$ from $\mathcal{H}^-$ (and counts the number of fermions modulo two). Thus we define $(-1)^F\psi = \psi$ for $\psi \in \mathcal{H}^+$, and $(-1)^F\chi = -\chi$ for $\chi \in \mathcal{H}^-$. The first basic condition which must be satisfied by the supersymmetry operators Q_i is that they each anticommute with $(-1)^F$:

$$(-1)^F Q_i + Q_i(-1)^F = 0. \tag{1}$$

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