J. DIFFERENTIAL GEOMETRY
17 (1982) 307–335

A REGULARITY THEORY FOR HARMONIC MAPS

RICHARD SCHOEN & KAREN UHLENBECK

0. Introduction

In this paper we develop a regularity theory for energy minimizing harmonic maps into Riemannian manifolds. Let u: $M^n \to N^k$ be a map between Riemannian manifolds of dimension n and k. It was shown by C. B. Morrey [17] in 1948 that if $n = 2$, then an energy minimizing harmonic map is Hölder continuous (and smooth if M and N are smooth). Since that time results have been found under special assumptions on N. Eells and Sampson [5] proved in 1963 that if N is compact and has nonpositive curvature, then every homotopy class of maps from a closed manifold M into N has a smooth harmonic representative. In the case where the image of the map is contained in a convex ball of N, there is a complete existence and regularity theory due to Hildebrandt and Widman [15] as well as Hildebrandt, Kaul and Widman [13]. Recently Giaquinta and Giusti obtained results for the case in which the image lies in a coordinate chart [9], [10].

In this paper we show that a bounded, energy minimizing map u: $M^n \to N^k$ is regular (in the interior) except for a closed set $\mathcal{S}$ of Hausdorff dimension at most $n - 3$. We also show $\mathcal{S}$ is discrete for $n = 3$. Moreover, we derive techniques (see Theorem IV) for lowering the dimension of $\mathcal{S}$ under the condition that certain smooth harmonic maps of spheres into N are trivial. This can be checked in some interesting cases, for example if N has nonpositive curvature or if the image of the map lies in a convex ball of N, we show $\mathcal{S} = \varnothing$ and any minimizing harmonic map into such a manifold is smooth. Using our methods, it is possible to reduce the dimension of $\mathcal{S}$ if N is a sphere or Lie group by studying harmonic spheres in N. Our methods work for functionals which are the energy plus lower order terms, and thus have direct bearing on the question of the existence of global Coulomb gauges in nonabelian gauge theories.

We point out that there is a strong historical precedent for partial regularity results in problems involving elliptic systems (see Almgren [1], De Giorgi [3],

Received January 26, 1982. Research of the first author was supported by the Sloan Foundation.