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ON THE CUSPIDAL SPECTRUM FOR FINITE VOLUME SYMMETRIC SPACES

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1. Introduction

Let $K \setminus G/\Gamma$ be a noncompact locally symmetric space of finite volume. Here G is a semisimple Lie group and Γ is an arithmetic subgroup. Moreover, K is a maximal compact subgroup.

If Δ is the Laplacian on $K \setminus G/\Gamma$, we consider Δ acting on the cuspidal functions $L^2_{\text{cusp}}(K \setminus G/\Gamma)$ in the sense of Langlands [14]. Our main result is the following:

Theorem 1.1. *Let $N(\lambda)$ be the number of linearly independent cuspidal eigenfunctions with eigenvalue less than λ. Then $N(\lambda)$ is finite for each fixed $\lambda > 0$.*

Moreover, one has the asymptotic upper bound:

$$\overline{\lim_{\lambda \to \infty}} \frac{N(\lambda)}{\lambda^{d/2}} \leqslant (4\pi)^{-d/2} \frac{\operatorname{vol}(K \setminus G/\Gamma)}{\Gamma(d/2 + 1)}. \tag{1.2}$$

Here d is the dimension of $K \setminus G/\Gamma$ and vol *denotes the volume. Also, $\Gamma(d/2 + 1)$ is the ordinary Gamma function.*

The fact that $N(\lambda)$ is finite for fixed $\lambda > 0$ was announced by Borel and Garland [2], [10].

If $G = \mathrm{SL}(2, R)$, then Theorem 1.1 has apparently been well known for some time. It certainly follows from the scattering theory of [15], although the explicit estimate is not stated there. Several authors [21] have given more detailed information for particular discrete subgroups Γ of $\mathrm{SL}(2, R)$. In the case $\Gamma = \mathrm{SL}(2, Z)$, equality holds in (1.2) and the limit on the left-hand side exists [15], [20].

When G is a real rank one, Gangolli and Warner [9] obtained the estimate $N(\lambda) \leqslant C\lambda^n$, for some C and n. However, their method did not give a good estimate of n.

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