J. DIFFERENTIAL GEOMETRY
16 (1981) 401–406

FOLIATED MANIFOLDS WITH FLAT BASIC CONNECTION

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1. Introduction and statement of results

Let $\mathfrak{F}$ be a smooth codimension-q foliation of a smooth manifold M. Let $T(M)$ denote the tangent bundle of M, and let $E \subset T(M)$ be the subbundle consisting of the vectors tangent to the leaves of $\mathfrak{F}$. Let $Q = T(M)/E$ be the normal bundle of $\mathfrak{F}$, and let $F(Q)$ be its frame bundle, a principal $GL(q, R)$ bundle. Recall that a connection on $F(Q)$ is said to be basic if the parallel translation which it defines along paths lying in a leaf of $\mathfrak{F}$ agrees with the "natural parallelism along the leaves" [3]. Equivalently, if π: $T(M) \to Q$ is the natural projection, and if $\Gamma(E)$, $\Gamma(Q)$, and $\mathfrak{X}(M)$ denote the space of smooth sections of the vector bundles E, Q, and $T(M)$ respectively, then the associated Koszul operator $\nabla: \mathfrak{X}(M) \times \Gamma(Q) \to \Gamma(Q)$ satisfies the condition that $\nabla_X Y = \pi([X, \tilde{Y}])$ for all $X \in \Gamma(E)$ and all $Y \in \Gamma(Q)$, where $\tilde{Y}$ is any vector field on M such that $\pi(\tilde{Y}) = Y$, and $[X, \tilde{Y}]$ denotes the usual Lie bracket of vector fields [2]. In the present work we study foliated manifolds supporting a flat basic connection, that is, a basic connection with vanishing curvature and torsion.

To begin, we have the following nonexistence result.

Theorem 1. *If M is compact with finite fundamental group, then M does not support a foliation with flat basic connection.*

As a corollary to the proof of Theorem 1, we will obtain

Corollary 1. *Let $(M, \mathfrak{F})$ be a foliated manifold with flat basic connection. If $H_1(M, Z) = 0$, then $\mathfrak{F}$ admits a transverse volume element; that is, $\mathfrak{F}$ is defined by a nowhere zero closed q-form on M, $q =$ codim$(\mathfrak{F})$.*

It is well-known (see, e.g., [6]) that the universal cover of an n-dimensional manifold supporting a complete flat linear connection is R^n where the lifted connection corresponds to the canonical linear connection on R^n. We generalize this codimension-n result to foliations of arbitrary codimension.

Communicated by R. A. Bott, June 24, 1980.