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THE EULER CYCLE OF A FOLIATION

ANTHONY PHILLIPS & DAVID STONE

Introduction

The Poincaré dual of the Euler class of (the tangent bundle of) a codimension-one foliation is a one-dimensional homology class. For a sufficiently smooth foliation the Euler class may be calculated locally by choosing a Riemannian metric on the underlying manifold. This determines a closed form (for a 2-dimensional foliation this is the curvature form of the induced connection on the tangent bundle to the foliation) which when integrated over any closed leaf gives its Euler characteristic (Gauss-Bonnet Theorem) and which in fact represents the Euler class of the foliation.

We present here an analogous construction for the Poincaré dual of the Euler class, but combinatorial rather than differentiable. The foliation now need only be of class $C^{0,1}$, i.e., tangent to a continuous field of hyperplanes. For the choice of a Riemannian metric we substitute the choice of a smooth triangulation in general position (in Thurston's sense) with respect to the foliation. We prove that this choice determines a locally computable 1-cycle representing the dual class. This will be the Euler cycle. The analogy with the differentiable construction is more than formal: the coefficient of a 1-simplex transverse to the foliation is equal to the "combinatorial curvature" (defined following Banchoff in §2) of the leaves of the foliation at the points where it intersects them.

The search for such a combinatorial expression was motivated by the representation, conjectured by Stiefel [9, p. 342], proved and reproved by Whitney, Cheeger, and Halperin-Toledo [5] for the duals of the Stiefel-Whitney classes of a manifold.

Consider then a smooth oriented n-manifold M carrying a transversely oriented $(n-1)$-dimensional foliation $\mathcal{F}$. We assume as mentioned above

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