J. DIFFERFNTIAL GEOMETRY
8 (1972) 331-334

AN INTEGRAL FORMULA

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The following results generalize in several directions a recent formula of Richard Kraft [3, Lemma 1] used in a problem of geometrical optics.

Theorem 1. *Let M be a compact n-manifold imbedded in a Euclidean $(n+1)$-space E^{n+1}, where $n > 1$. For each $x \in M$, let $e = e(x)$ be the outward unit normal, $r = |x|$, and $p = p(x) = x \cdot e$, the support function. Also let σ denote the element of n-volume. Then*

$$\frac{1}{V_n} \int_M \frac{px}{r^{n+2}} \sigma = \begin{cases} 0 & \text{if } 0 \notin M\,, \\ -e(0) & \text{if } 0 \in M\,, \end{cases}$$

where $V_n = \pi^{n/2}/\Gamma(\frac{1}{2}n + 1)$ is the volume of the unit n-ball.

Our proof will be based on two formal lemmas. We shall denote by $[v_1, \cdots, v_n]$ the cross (vector) product of n vectors in E^{n+1}, assumed oriented. As usual, we extend this alternating multilinear function to vectors with differential form coefficients by

$$[\alpha_1 v_1, \cdots, \alpha_n v_n] = (\alpha_1 \wedge \cdots \wedge \alpha_n)[v_1, \cdots, v_n]\ .$$

We refer to Flanders [1, pp. 43, 149] and [2] for this formalism.

Lemma 1. *On M we have*

$$n(x \cdot dx) \wedge [x, dx, \cdots, dx] = r^2[dx, \cdots, dx] - n!\, px\sigma\ .$$

Proof. We shall give more detail than is really necessary, because the probability of an error in sign is high in calculations of this type.

Let $e_1, \cdots, e_n$ be a moving orthonormal frame on M, so

$$x = p_i e_i + pe\ , \qquad dx = \sigma_i e_i\ ,$$

where the σ_i are one-forms, and repeated indices are summed. Note that $\sigma_1 \wedge \cdots \wedge \sigma_n = \sigma$ is the volume element on M. We take the e_i so that $e_1, \cdots, e_n, e$ is a right-handed frame for E^{n+1}. Then $[e_1, \cdots, e_n] = e$. We also note for future reference that

$$[e, e_1, \cdots, \hat{e}_i, \cdots, e_n] = (-1)^i e_i\ ,$$

Received May 3, 1972.