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On Linear Integro-Differential Equations in a Banach Space

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Introduction

In this paper we study a linear Volterra integro-differential equation of the form

$$\text{(E)} \qquad \frac{d}{dt}u(t)=Au(t)+\int_0^t B(t-s)u(s)ds+f(t) \qquad \text{for} \quad t>0,\ u(0)=x\ ,$$

in a Banach space X with norm $\|\cdot\|$. Here $f\colon R_+=[0, \infty)\rightarrow X$ is continuous and A is the infinitesimal generator of a semi-group of class (C_0) on X. For each $t\in R_+$ $B(t)$ is a (in general unbounded) linear operator with domain dense in X. Let $B(X)$ denote the set of all bounded linear operators from X into itself.

It is well known [2] that on a finite dimensional space $X=R^n$ (the n-dimensional space of column vectors with the usual norm $|\cdot|$),

$$\text{(0.1)} \qquad u(t)=U(t)x+\int_0^t U(t-s)f(s)ds \qquad \text{for} \quad t>0\ ,$$

is a unique solution of (E) for $x\in X$. In this case A and $B(t)$ are $n\times n$ matrices, $B(t)$ is a locally integrable function on R_+ and the $n\times n$ matrix function $U(t)$ is the solution of the equation

$$\frac{d}{dt}U(t)=AU(t)+\int_0^t B(t-s)U(s)ds=U(t)A+\int_0^t U(t-s)B(s)ds \qquad \text{for} \quad t>0\ ,$$
$$U(0)=I \text{ (the identity matrix).}$$

In a general Banach space X, it is also known [5, 12] that if $\{B(t); t\in R_+\}$ is in $B(X)$ and $B(t)x\colon R_+\rightarrow X$ is continuous for each $x\in X$, then there exists a one-parameter family $\{U(t); t\in R_+\}$ in $B(X)$ which satisfies the following two equations

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