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On a Differential Equation Characterizing a Riemannian Structure of a Manifold

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It often happens that the existence of a function on a riemannian manifold satisfying some condition gives informations about the topological, differentiable or riemannian structure of the manifold. In fact, in 1962, Obata characterized the euclidean sphere of radius $1/\sqrt{k}$ as the only complete riemannian manifold which has a nontrivial solution for the differential equation

$$\text{Hess}\, f + kfg = 0 \tag{1}$$

with a positive constant k, where $\text{Hess}\, f$ is a symmetric (0, 2)-tensor called the hessian of f defined by $(\text{Hess}\, f)(X, Y) = (\nabla_X df)(Y) = XYf - (\nabla_X Y)f$ for any vector fields X and Y, and g is the metric tensor: that is

THEOREM A (Obata [3, 4]). *Let $k>0$. For a C^∞ complete riemannian manifold (M, g) of dimension $n(\geqq 2)$, there is a C^∞ nontrivial function f on M satisfying* (1), *if and only if (M, g) is isometric to the euclidean n-sphere $(S^n, (1/k)g_0)$ of radius $1/\sqrt{k}$, where g_0 denotes the canonical metric on S^n with constant curvature* 1.

Also there is a work by Tanno [5] in which he investigated effects of some differential equations of order three on riemannian and kählerian manifolds.

In this article we give necessary and sufficient conditions for the existence of a nontrivial function f on (M, g) which satisfies (1) with a nonpositive constant k. A manifold is assumed to be of C^∞ and connected, unless otherwise indicated. Also all tensors (including functions, vector fields, etc.) are assumed to be C^∞, unless otherwise indicated.

The case $k=0$ is reduced to the following trivial theorem:

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