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On Normal Integral Bases

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Introduction

Let k be a number field, and K/k a finite Galois extension with Galois group G. Let $\mathfrak{o}_k$ and $\mathfrak{o}_K$ be the rings of integers in k and K. We denote by $\mathfrak{o}_k G$ the group ring of G over $\mathfrak{o}_k$. $\mathfrak{o}_K$ can be regarded as an $\mathfrak{o}_k G$-module by the action $r\cdot\alpha=\sum_{s\in G} a_s s\alpha$ for $\alpha\in\mathfrak{o}_K$, $r=\sum_{s\in G} a_s s\in\mathfrak{o}_k G$. These notations will be used throughout this paper. K/k is said to have a *normal integral basis* (abbr. n.i.b.) when there is an element $\alpha\in\mathfrak{o}_K$ such that $\{s\alpha\}_{s\in G}$ is a relative integral basis of K/k, and α is called a *generator* of this basis. It is known that a finite Galois extension with n.i.b. is tamely ramified ([4], Chapter 9, Theorem (1, 2)).

In case where k is the field $\boldsymbol{Q}$ of rational numbers, every tamely ramified abelian field has an n.i.b. (Hilbert-Speiser), so that when $k=\boldsymbol{Q}$ and G is abelian, K/k has an n.i.b. if and only if K/k is tamely ramified ([4], Chapter 9, Theorem (3, 4)). Furthermore, Fröhlich [2] has given a necessary and sufficient condition for K/k to have an n.i.b., when K/k is a Kummer extension. On the other hand, Okutsu [8] has shown that when $k=\boldsymbol{Q}(\zeta_l)$, $\zeta_l=\exp(2\pi i/l)$, l: odd prime, and $K=k(\sqrt[l]{a})$, $a\in\boldsymbol{Z}$, K/k has always a relative integral basis and given an explicit form of this basis. After preparations in §1, giving in particular a more precise form to the results of [2], we shall apply them in §2 to obtain a necessary and sufficient condition for K/k to have an n.i.b. for the case where k and K are as in [8]. We shall also give explicitly a generator of n.i.b. when this exists. In the final §3, we shall construct many examples of normal extensions K/k with n.i.b.'s where $k\neq\boldsymbol{Q}$, and K/k are tamely ramified. We shall also mention an example of such K/k without n.i.b..

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