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On the Mean-Square for the Approximate Functional Equation of the Riemann Zeta-Function

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§1. Introduction.

Let $d(n)$ be the number of positive divisors of n, and γ the Euler constant. The problem of estimating the quantity

$$\Delta(x)=\sum_{n\leqq x}{}' d(n)-x(\log x+2\gamma-1)-1/4$$

is called the Dirichlet divisor problem, where the symbol $\sum'$ indicates that the last term is to be halved if x is an integer. G. F. Voronoi [9] proved two remarkable formulas concerning $\Delta(x)$. Besides giving an explicit expression for $\Delta(x)$, he (see also (2.3) of [2]) proved

$$\int_2^T \Delta(x)dx=4^{-1}T+(2\sqrt{2}\,\pi^2)^{-1}T^{3/4}\sum_{n=1}^{\infty} d(n)n^{-5/4}\sin(4\pi\sqrt{nT}-4^{-1}\pi)+O(1)\,. \quad (1.1)$$

The sharp estimate of the error term is to be noted.

Let $\zeta(s)$ be the Riemann zeta-function, and for $T\geqq 2$, let

$$E(T)=\int_0^T |\zeta(1/2+it)|^2dt-T\log(T/2\pi)-(2\gamma-1)T\,, \quad (1.2)$$

the error term in the mean-square formula for $\zeta(s)$. J. L. Hafner and A. Ivić [2] established the analogue of (1.1) for $E(T)$:

$$\int_2^T E(t)dt=\pi T+2^{-1}(2T/\pi)^{3/4}\sum_{n=1}^{\infty}(-1)^n d(n)n^{-5/4}\sin(4\pi(nT/2\pi)^{1/2}-4^{-1}\pi)$$
$$+O(T^{2/3}\log T) \qquad \text{(see (2.5) of [2])}\,. \quad (1.3)$$

We note that apart from the factor $(-1)^n$ the series in (1.3) is the same as the one in

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