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On the Algebraic Independence of Certain Numbers Connected with the Exponential and the Elliptic Functions

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Introduction

It has been conjectured in transcendental number theory that π and $\log 2$ are algebraically independent. It has also been conjectured that at least one of the numbers $\sum_{n=0}^{\infty} 2^{-n^2}$ and $\sum_{n=0}^{\infty} (-1)^n 2^{-n^2}$ is transcendental. Though no one has ever proved these conjectures, the authors have proved the following

PROPOSITION. *At least two of the numers*

$$\pi\ , \quad \log 2\ , \quad \sum_{n=0}^{\infty} 2^{-n^2}\ , \quad \sum_{n=0}^{\infty} (-1)^n 2^{-n^2}$$

are algebraically independent over $\boldsymbol{Q}$. (This is a special case of Example 2.1, §1.)

Let x be a transcendental number, and let κ be a real number $\geqq 2$. We shall say that x is of *transcendence type* $\leqq \kappa$ if there exists a constant $c>0$ depending only on x and κ such that

$$\log |P(x)| \geqq -c(\deg P + \log H(P))^{\kappa}$$

for all non-trivial polynomials P in $\boldsymbol{Z}[X]$. Here, $\deg P$ denotes the degree of P, and $H(P)$ denotes the height of P, i.e. the maximum of the absolute values of the coefficients of P.

The idea of transcendence type was introduced by Lang in his book [4]. For example, it follows from Fel'dman's result [2, Theorem 4] that

(1) $\qquad \pi$ is of transcendence type $\leqq 2+\varepsilon$, for every $\varepsilon>0$.

This is a well-known result in transcendental number theory.

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