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On the First Eigenvalue of the p-Laplacian in a Riemannian Manifold

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1. Introduction and results.

Let Ω be a bounded domain in a Riemannian manifold (M, g) of dimension m. We consider the following Dirichlet problem:

$$(1)\qquad \begin{aligned} \Delta_p u+\lambda|u|^{p-2}u&=0 \quad \text{in } \Omega\,, \\ u&=0 \quad \text{on } \partial\Omega\,, \end{aligned}$$

where $\Delta_p u=\operatorname{div}(|\nabla u|_g^{p-2}\nabla u)$ is the p-Laplacian with $1<p<\infty$. In local coordinates,

$$\Delta_p u=\frac{1}{\sqrt{\det(g_{ij})}}\sum_{i,j=1}^{m}\frac{\partial}{\partial x^i}\left(\sqrt{\det(g_{ij})}g^{ij}|\nabla u|^{p-2}\frac{\partial u}{\partial x^j}\right),$$

where $|\nabla u|^2=|\nabla u|_g^2=\sum_{ij}g^{ij}(\partial u/\partial x^i)(\partial u/\partial x^j)$, and $(g^{ij})=(g_{ij})^{-1}$. The *first eigenvalue* $\lambda_{1,p}(\Omega)$ *of the p-Laplacian* is defined as the least real number λ for which the Dirichlet problem (1) has a nontrivial solution $u\in W_0^{1,p}(\Omega)$. Here the Sobolev space $W_0^{1,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the Sobolev norm $\|u\|_{1,p}=\{\int_\Omega(|u|^p+|\nabla u|^p)dv_g\}^{1/p}$. It can be also characterized by

$$(2)\qquad \lambda_{1,p}(\Omega)=\inf_{u\neq 0}\frac{\int_\Omega|\nabla u|^p dv_g}{\int_\Omega|u|^p dv_g}\,,$$

where u runs over $W_0^{1,p}(\Omega)$ and dv_g denotes the volume element of M. We would like to estimate the $\lambda_{1,p}(\Omega)$. For the case $p=2$, there have been several results, such as the Faber-Krahn inequality [1], the Cheeger inequality [2], and the Cheng inequality [3]. The purpose of this paper is to give inequalities for their p-Laplacian analogue. More precisely we show the following theorems.

THEOREM 1 (the Faber-Krahn type inequality). *Let M_k be a complete simply connected Riemannian manifold of constant sectional curvature κ. Let B be the geodesic*

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