TOKYO J. MATH.
VOL. 24, NO. 1, 2001

Some Character Sums and Gauss Sums over $G_2(q)$

Naomichi SAITO and Ken-ichi SHINODA

Sophia University

0. Introduction.

Following the work of T. Kondo [11] for $GL_n(q)$, we defined a Gauss sum for a finite reductive group defined over $\mathbf{F}_q$ associated with a modular representation over the same characteristic and a complex ordinary representation in [15]. When the ordinary representation is irreducible, the determination of the value of a Gauss sum is equivalent to that of the trace, τ_W (see (2.1) for details). Fixing the modular representation, we have shown that the values of τ_W are canonically determined for the generalized characters of Deligne-Lusztig and gave the values for finite classical groups with their canonical representation using Kloosterman sums and unitary Kloosterman sums. Moreover as an example we have determined the values of Gauss sums over $Sp(4, q)$, with q odd, associated with every complex irreducible representation and with the canonical modular representation.

On the other hand, in a series of papers starting with [10], D. S. Kim, I. Lee, and K. Park have considered a Gauss sum over a finite reductive group, when the complex representation is one dimensional and factored through the determinant of the modular representation. In particular Lee and Park, [12], have determined the Gauss sum for the Chevalley group of type G_2 over $\mathbf{F}_q$, $G_2(q)$, associated with the irreducible 7-dimensional modular representation and the trivial (ordinary) representation.

The purpose of this paper is to give an explicit expression for Gauss sum over $G_2(q)$ associated with every unipotent character and with the 7-dimensional modular representation, applying the method developed in [15]. In particular we give another proof for the theorem obtained by Lee and Park cited above. On the way we also have the Gauss sums associated with $\varepsilon(\mathbf{T})R_{\mathbf{T},\theta}$, when they are irreducible.

Interestingly enough these sums corresponding to unipotent characters are written by using one character sum. In general, it seems that there is a character sum associated with each geometric conjugacy class $\{(\mathbf{T}, \theta)\}$ (cf. [7]), and that if an irreducible character ζ appears as a component of $R_{\mathbf{T},\theta}$, the corresponding Gauss sum $w(\zeta)$ (cf. (2.1)) can be expressed using the character sum.

In our case, we give five relations for the character sums associated with $\{R_{\mathbf{T},1}\}$, and section 1 is devoted to establish such relations between certain character sums. Applying the

Received March 23, 2000