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Existence and Regularity Results for Harmonic Maps with Potential

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1. Introduction.

Let (M, g) and (N, h) be Riemannian m- and n- manifolds, G a smooth function on N. For a bounded domain $\Omega \subset M$ and a map $u : M \to N$ we define *the energy functional with the potential G on Ω* :

$$E_G(u; \Omega) = \int_\Omega [e(u) - G(u)]d\mu\,, \tag{1.1}$$

where $e(u)$ and $d\mu$ are the standard energy density and the volume element on M. Using local coordinate systems $(x^1, \cdots, x^m)$ and $(u^1, \cdots, u^n)$ on M and N respectively, we can write

$$E_G(u; \Omega) = \int_\Omega \left[\frac{1}{2}g^{\alpha\beta}(x)h_{ij}(u)D_\alpha u^i D_\beta u^j - G(u(x))\right]\sqrt{g}dx\,, \tag{1.2}$$

where $(g^{\alpha\beta}(x)) = (g_{\alpha\beta}(x))^{-1}$, $g = \det(g_{\alpha\beta}(x))$ and $D_\alpha = \partial/\partial x^\alpha$. The Euler-Lagrange equation of E_G is given as

$$\tau(u) + \nabla G = 0\,, \tag{1.3}$$

where $\tau(u)$ denotes the tension field of u. In local

$$(\tau(u))^i = \frac{1}{\sqrt{g}}D_\alpha\{\sqrt{g}g^{\alpha\beta}D_\beta u^i\} + g^{\alpha\beta}\Gamma^i_{jk}D_\alpha u^j D_\beta u^k\,.$$

A solution $u : \Omega \to N$ of (1.3) is called to be *a harmonic map with potential G*.

The equations of type (1.3) appear also in some physical contexts. Let $\Omega \subset \mathbf{R}^m$, $N = S^2 = \{(x, y, z) \in \mathbf{R}^3; x^2 + y^2 + z^2 = 1\}$ and $G(u) = (u, H) = u^1H^1 + \cdots + u^nH^n$ for some constant vector $H \in \mathbf{R}^3$, then the equation (1.3) becomes

$$\Delta u + u|Du|^2 - (H, u)u + H = 0 \tag{1.4}$$

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