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On The Unit Group of The Group Ring $\mathbb{Z}[G]$

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Introduction.

Let G be a commutative group. A formula on the torsion free rank of $\mathbb{Z}[G]$ is given by Higman ([2, Theorem 13.5]). We think about a case where G is a finite commutative group. Then we can define a fundamental system of units in $\mathbb{Z}[G]$ (See Definition 2.2.). We consider the following problem.

PROBLEM A. Given a finite commutative group G, find a specific fundamental system of units in $\mathbb{Z}[G]$.

This is a difficult problem. For example, if G is cyclic of prime order p, then Problem A is equivalent to the problem of find a specific fundamental system of units of the subgroup of $\mathbb{Z}[\zeta]^\times$ consisting of all units which are congruent to 1 modulo $\zeta - 1$, where ζ be a primitive p-th root of unity. Therefore we consider the weaker next problem.

PROBLEM B. Given a finite commutative group G, find a specific system of r independent units of infinite order in $\mathbb{Z}[G]$ or, equivalently, a system of independent units of infinite order which generates a subgroup of finite index.

In the case where G is a cyclic group, an independence system of units in $\mathbb{Z}[G]$ is given by Bass ([1], [2]). In this article, we consider the elementary p-group case $G = (\mathbb{Z}/p)^n$, and we give the direct product decomposition of $\mathbb{Z}[G]^\times$ induced by the structure of the unit group scheme $U(G)$.

ASSERTION 1 (cf. Lemma 2.3). *Let $G = (\mathbb{Z}/p)^n$ and let ζ be a primitive p-th root of unity. We put $\lambda = \zeta - 1$. Then*

$$\mathbb{Z}[G]^\times \xrightarrow{\sim} \{\pm 1\} \times \prod_{i=1}^{n} U_i^{\binom{n}{i}},$$

where $U_i := \{\tilde{u} \in (\mathbb{Z}[\zeta]^{\otimes i})^\times | \tilde{u} \equiv 1^{\otimes i} \bmod \lambda^{\otimes i}\}$.

Moreover we construct an independent system of finite index of the unit group $\mathbb{Z}[G]^\times$ when $G = \mathbb{Z}/p \times \mathbb{Z}/p$.

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