"Hasse principle" for $SL_n(D)$

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1. Notation and results. Extending the usage of language in Galois cohomology, T. Ono defined "Hasse principle" for any group G (cf. [2]). We know that the Hasse principle holds for $G =$ abelian, dihedral, quatenion, $PSL_2(\mathbf{Z})$, $PSL_2(\mathbf{F}_p)$ (cf. [2]), free groups ([3]), symmetric groups and alternating groups ([4]).
Let D be an Euclidean domain(for examples $D = \mathbf{Z}$, $D = \mathbf{F}_p$). Put $\varepsilon = (-1)^{n-1}$ and we define in $SL_n(D)$

$$S = \begin{pmatrix} 0 & \dots & \dots & 0 & \varepsilon \\ 1 & \ddots & & & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix},$$

$$T_\mu = \begin{pmatrix} 1 & \mu & 0 & \dots & 0 \\ 0 & \ddots & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}, \ T = T_1.$$

Then $SL_n(\mathbf{Z})$ is generated by S and T (cf.[1]). Similarly using Eucledian algorithm we can prove that $SL_n(D)$ is generated by S and $\{T_\mu \mid \mu \in D\}$. Using this fact, T. Ono proved that $SL_2(D)$ enjoys the Hasse principle (unpublished). In this paper, we shall prove more generally the following

Theorem. *For any natural number n, $SL_n(D)$ and $PSL_n(D)$ enjoy the Hasse principle.*

M. Mazur [5] noticed that $f(x)$ is a cocycle iff $g(x) = f(x)x$ is an endomorphism of G, $f(x)$ is a local coboundary iff $g(x) \sim x$ (conjugate in G) for each $x \in G$ and $f(x)$ is a global coboundary iff $g(x)$ is an inner automorphism. Therefore "Hasse principle" is equivalent to say that "any endomorphism of G which satisfies $g(x) \sim x$ for each $x \in G$ must be an inner automorphism".

2. Proof of the theorem. Let $g(x)$ be an endomorphism which satisfies $g(x) \sim x$ for each $x \in G$. We may assume $g(S) = S$, $g(T) = M^{-1}TM$ where

$$M = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \\ \vdots & \vdots & & \vdots \\ g_1 & g_2 & \dots & g_n \end{pmatrix} = \begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \\ \vdots \\ \boldsymbol{g} \end{pmatrix},$$

$$M^{-1} = \begin{pmatrix} x_1 & \\ x_2 & \\ \vdots & * \\ x_n & \end{pmatrix}.$$

Then

$$g(T) = E + M^{-1}E_{12}M = \begin{pmatrix} 1 + x_1b_1 & x_1b_2 & \dots & x_1b_n \\ x_2b_1 & 1 + x_2b_2 & \dots & x_2b_n \\ \vdots & \vdots & & \vdots \\ x_nb_1 & x_nb_2 & \dots & 1 + x_nb_n \end{pmatrix}.$$

where E_{ij} is the matrix unit whose ij-element is 1 and the other elements are 0. Put

$$\tilde{M} = \begin{pmatrix} \boldsymbol{v}_1 \\ \boldsymbol{b} \\ \boldsymbol{v}_3 \\ \boldsymbol{v}_4 \\ \vdots \\ \boldsymbol{v}_n \end{pmatrix} = \begin{pmatrix} b_2 & b_3 & b_4 & \dots & b_n & \varepsilon b_1 \\ b_1 & b_2 & b_3 & \dots & b_{n-1} & b_n \\ \varepsilon b_n & b_1 & b_2 & \dots & b_{n-2} & b_{n-1} \\ \varepsilon b_{n-1} & \varepsilon b_n & b_1 & \dots & b_{n-3} & b_{n-2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \varepsilon b_3 & \varepsilon b_4 & \varepsilon b_5 & \dots & b_1 & b_2 \end{pmatrix}. \tag{1}$$

Then after a little calculation we have

$$|xE - Sg(T)| = x^n - \sum_{i=1}^{n-1}(x_1b_{i+1} + x_2b_{i+2} + \dots + x_{n-i}b_n + \varepsilon x_{n-i+1}b_1 + \dots + \varepsilon x_nb_i)x^{n-i} - \varepsilon =$$