A condition of quasiconformal extendability

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Recently, Heinonen and Koskela showed, as a corollary of their deep result, the following extension theorem.

Proposition 1 ([3], 4.2 Theorem). *Suppose that f is a quasiconformal map of the complement of a closed set E in $\mathbf{R}^n$ into $\mathbf{R}^n$, $n \geq 2$, and suppose that each point $x \in E$ has the following property: there is a sequence of radii r_j, $r_j \to 0$ as $j \to \infty$, such that the annular region $B(x, ar_j) - B(x, r_j/a)$ does not meet E for some $a > 1$ independent of j. Then f has a quasiconformal extension to $\hat{\mathbf{R}}^n = \mathbf{R}^n \cup \{\infty\}$. Moreover, the dilatation of the extension agrees with the dilataiton of f.*

There, they remarked that this result may be new even for conformal maps in the plane. So it is noteworthy to give a different proof of a more general extension theorem on 2-dimensional quasiconformal maps of the plane based on some classical results in the function theory.

We begin with the following definition, which weakens the condition in the above theorem to a conformally invariant one.

Definition. We say that a closed set E in the complex plane is *annularly coarse* if each point $x \in E$ has the following property: there is a sequence of mutually disjoint nested annuli $\{R_k\}_{k=1}^{\infty}$, $R_k \cap E = \phi$, such that the modulus $m(R_k)$ of R_k satisfies

$$m(R_k) \geq c$$

with a positive c. Here we say that a sequence of annuli $\{R_k\}_{k=1}^{\infty}$ is *nested* if every R_k $(k > 1)$ separates R_{k-1} from x.

Also note that the positive constant c can depend on x.

Now we will prove the following

Theorem 2. *Suppose that f is a quasiconformal map of the complement of a closed set E in the complex plane $\mathbf{C}$ into $\mathbf{C}$ and suppose that E is annularly coarse. Then f has a quasiconformal extension to $\hat{\mathbf{C}}$. Moreover, the dilatation of the extension agrees with the dilataiton of f.*

1. Known facts and basic lemmas.

In 2-dimensional case, we have the following

Proposition 3. *Let E be a compact set in $\mathbf{C}$. Then the following conditions are mutually equivalent.*

1) *Every conformal map of $D = \mathbf{C} - E$ is the restriction of a Möbius transformaiton.*
2) *Every quasiconformal map of $D = \hat{\mathbf{C}} - E$ has a quasiconformal extension to the whole $\hat{\mathbf{C}}$.*
3) *For every relatively compact neighborhood U of E, every quasiconformal map of $U - E$ has a quasiconformal extension to U.*

Proof. First assume the condition 1) and take any quasiconformal map f of $D = \mathbf{C} - E$. Here we may assume that $f(\infty) = \infty$. Let μ be the Beltrami coefficient of f^{-1} on $f(\mathbf{C} - E)$. Set $\mu = 0$ on $\mathbf{C} - f(\mathbf{C} - E)$, and we have a quasiconformal map g of $\hat{\mathbf{C}}$ with the complex dilatation μ (cf. [2] and [4]). Then, $g \circ f$ has vanishing complex dilatation on $\mathbf{C} - E$, and hence the assumption implies that it is a Möbius transformation T. Thus f can be extended a quasiconformal map $g^{-1} \circ T$ of the whole $\hat{\mathbf{C}}$.

Next assume the condition 2) and take a relatively compact neighborhood U of E and a quasiconformal map f of $U - E$ arbitrarily. Since E is compact, the famous extention theorem ([6] II Theorem 8.1) gives a neighborhood V of E in U and a quasiconformal map g of $\hat{\mathbf{C}} - E$ which coincides with f on $V - E$. Then the assumption implies that g can be extended to a quasiconformal map of $\hat{\mathbf{C}}$, which clearly gives a quasiconformal extension of f to U.

Finally, assume the condition 3) and take any conformal map f of $D = \mathbf{C} - E$. Then f can be extended to a quasiconformal map g of $\mathbf{C}$. Hence if E has vanishing area, then this g is actually conformal, and hence is a Möbius transformation. If not, consider the extremal (horizontal) slit map h of $\mathbf{C} - E$. Then h should be extended a quasiconformal map of $\mathbf{C}$. But this is impossible, for $\mathbf{C} - f(\mathbf{C} - E)$ has

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