Homotopy groups of compact exceptional Lie groups

By Hideyuki KACHI*) and Mamoru MIMURA**)

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We consider homotopy groups of compact, connected, simply connected simple exceptional Lie groups G; they are classified as G_2, F_4, E_6, E_7 and E_8. By the Hopf theorem we have

$$H^*(G;Q) \cong \Lambda(x_1, x_2, \cdots, x_\ell),$$

where ℓ is the rank of G, $\deg x_i$ is odd and $\sum_{i=1}^{\ell} \deg x_i$ is the dimension of G. Let X be the direct product of spheres of dimensions $\deg x_1, \cdots, \deg x_\ell$, then, from Serre's $\mathcal{C}$-theory [10], $\pi_i(G)$ are $\mathcal{C}$-isomorphic to $\pi_i(X)$ for all i, where $\mathcal{C}$ is the class of finite abelian groups. Therefore the rank of $\pi_q(G)$ is equal to the number of such i that $\deg x_i$ is equal to q, particularly if q is even, then $\pi_q(G)$ is finite. It is a classical fact that $\pi_2(G) = 0$ and $\pi_3(G) \cong Z$.

According to Bott-Samelson [1], we have

$$\begin{aligned} &\pi_i(E_6) = 0 \text{ for } 4 \le i \le 8, && \pi_9(E_6) \cong Z, \\ &\pi_i(E_7) = 0 \text{ for } 4 \le i \le 10, && \pi_{11}(E_7) \cong Z, \\ &\pi_i(E_8) = 0 \text{ for } 4 \le i \le 14, && \pi_{15}(E_8) \cong Z. \end{aligned}$$

For $i \le 23$, the homotopy groups $\pi_i(G_2)$ and $\pi_i(F_4)$ are determined in [7] by the second author. For $G = E_6$, E_7 and E_8, the 2-primary components of homotopy group $\pi_i(G)$ are determined in [5] by the first author for $i \le 22$, $i \le 25$ and $i \le 28$ respectively.

The purpose of this paper is to calculate $\pi_i(G:3)$ the 3-primary components of homotopy groups $\pi_i(G)$ by making use of usual methods, such as the killing-homotopy method [2] and homotopy exact sequences associated with various fibrations.

Let $B_1(2n-1, 2n+3)$ be a S^{2n-1}-bundle over S^{2n+3} with the characteristic element $\alpha_1(2n-1) \in \pi_{2n+2}(S^{2n-1} : 3)$ so that $H^*(B_1(2n-1, 2n+3); Z_3) \cong \Lambda(x_{2n-1}, x_{2n+3})$, where $x_{2n+3} = \wp^1 x_{2n-1}$. Let $B_2(2n-1, 2n+7)$ be a S^{2n-1}-bundle over S^{2n+7} with the characteristic element $\alpha_2(2n-1) \in \pi_{2n+6}(S^{2n-1}:3)$ so that $H^*(B_2(2n-1, 2n+7); Z_3) \cong \Lambda(x_{2n-1}, x_{2n+7})$, where $x_{2n+7} = \Phi x_{2n-1}$; Φ is the secondary cohomology operation associated with the relation $\wp^1\beta\wp^1 - \beta\wp^2 - \wp^2\beta = 0$.

Homotopy group of G_2. It is known [7] that G_2 is 3-equivalent to $B_2(3, 11)$, and so $\pi_i(G_2)$ are $\mathcal{C}_3$-isomorphic to $\pi_i(B_2(3, 11))$, where $\mathcal{C}_3$ is the class of finite abelian groups without 3 torsion. Thus $\pi_i(G_2 : 3)$ can be calculated from the homotopy exact sequence

$$\begin{aligned} \cdots \to \pi_{i+1}(S^{11}) \to \pi_i(S^3) \to \pi_i(B_2(3, 11)) \\ \to \pi_i(S^{11}) \to \pi_{i-1}(S^3) \to \cdots \end{aligned}$$

associated with the bundle $B_2(3, 11)/S^3 = S^{11}$.

Homotopy group of F_4. Let K be a finite complex constructed by Harper in [3] such that

$$H^*(K; Z_3) \cong Z_3[x_8]/(x_8^3) \otimes \Lambda(x_3, x_7),$$

where $x_7 = \wp^1 x_3$ and $x_8 = \beta x_7$. Then, according to Harper [3], F_4 is 3-equivalent to $K \times B_1(11, 15)$.

We consider the 3-connective fiber space $\tilde{F}_4$ over F_4 by killing $\pi_3(F_4)$ so that $\pi_i(\tilde{F}_4) \cong \pi_i(F_4)$ for $i \ge 4$. Then we have that $\tilde{F}_4$ is 3-equivalent to $\tilde{K} \times B_1(11, 15)$, where $\tilde{K}$ is the 3-connective fiber space over K and $H^*(\tilde{K}; Z_3) \cong Z_3[y_{18}] \otimes \Lambda(y_{19}, y_{23})$, where $y_{19} = \beta y_{18}$ and $y_{23} = \wp^1\beta y_{18}$. Now we approximate $\tilde{K}$ by a finite cell complex $K' = S^{18} \cup_{3\iota_{18}} e^{19} \cup e^{23}$ such that $\pi_i(\tilde{K})$ is $\mathcal{C}_3$-isomorphic to $\pi_i(K')$ for $i \le 34$.

Then we have the homotopy exact sequence

$$\begin{aligned} \cdots \to \pi_i(S^{22}) \xrightarrow{f_*} \pi_i(S^{18} \cup_{3\iota_{18}} e^{19}) \to \pi_i(K') \\ \to \pi_{i-1}(S^{22}) \xrightarrow{f_*} \pi_{i-1}(S^{18} \cup_{3\iota_{18}} e^{19}) \to \cdots \end{aligned}$$

for $i \le 34$, where $f : S^{22} \to S^{18} \cup_{3\iota_{18}} e^{19}$ is the attaching map for e^{23}. Thus we can obtain $\pi_i(F_4 : 3)$ for $i \le 34$. (Recall that $\pi_i(B_1(11, 15)$ is calculated in [9]).

Homotopy group of E_6. According to Harris [4], E_6 is 3 equivalent to $F_4 \times B_2(9, 17)$. Thus $\pi_i(E_6 : 3)$ can be read off from those of F_4 and $B_2(9, 17)$, which is calculated from the homotopy exact sequence associated with the bundle $B_2(9, 17)/S^9 = S^{17}$.

Homotopy group of E_8. The 3-connective fiber space $\tilde{E}_8$ over E_8 has the following mod 3 co-

*) Department of Mathematical Sciences, Shinshu University, 3-1-1 Asahi, Matsumoto, Nagano 390–8621.
**) Department of Mathematics, Okayama University, 3-1-1 Tsushima-Naka, Okayama 700–8530.