A note on the Rankin-Selberg method for Siegel cusp forms of genus 2

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The purpose of this note is to give an explicit relation between certain Dirichlet series and spinor zeta functions attached to Siegel cusp forms of genus 2; a part of results in [7] is generalized to the case of *any level*. Thereby we point out that the method of [7] to study spinor zeta functions is applicable to higher levels.

1. Notations. We use standard notations, found in [2]. We let $\Gamma_2 := \mathrm{Sp}_2(\mathbf{Z})$ be integral symplectic 4×4-matrices and Γ_1 be the elliptic full modular group. We set

$$\Gamma_g(N) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g \middle|\, C \equiv O (\mathrm{mod}\ N) \right\}.$$

where A, B, C, D are $g\times g$-matrices. We let $\Gamma_1^J(N)$ be the semi direct product of $\Gamma_1(N)$ and $\mathbf{Z}^2$, which is called the *Jacobi group of level N*.

$\mathcal{H}_g$ denotes the Siegel upper half space of genus g consisting of complex $g\times g$-matrices with positive definite imaginary part. We often write

$$Z = X + iY = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in \mathcal{H}_2.$$

Let k be an integer > 2 and χ be a Dirichlet character modulo N. We write $S_k(N,\chi)$ for the space of holomorphic cusp forms on $\mathcal{H}_2$ of weight k and character χ with respect to $\Gamma_2(N)$, and $J_{k,l}^{\mathrm{cusp}}(N,\chi)$ for the space of holomorphic Jacobi cusp forms on $\mathcal{H}_1\times\mathbf{C}$ of weight k, character χ and index l with respect to $\Gamma_1^J(N)$. The Petersson inner product on these spaces are normalized by

$$\langle F, G\rangle_N := \int_{\Gamma_2(N)\backslash\mathcal{H}_2} F(Z)\bar{G}(Z)\,|Y|^{k-3}\,dX\,dY$$
$$(F, G \in S_k(N,\chi),\ Z = X+iY \in \mathcal{H}_2),$$
$$\langle \phi, \psi\rangle_N := \int_{\Gamma_1^J(N)\backslash\mathcal{H}_1\times\mathbf{C}} \phi(\tau,z)\,\bar{\psi}(\tau,z) \times \exp\left(-\frac{4\pi l y^2}{v}\right) v^{k-3} du\,dv\,dx\,dy$$
$$(\phi,\psi \in J_{k,l}^{\mathrm{cusp}}(N,\chi),\ \tau = u+iv \in \mathcal{H}_1,\ z = x+iy \in \mathbf{C}).$$

We write simply $\mathbf{e}(*)$ for $\exp(2\pi i *)$.

2. Statement of Result.

Definition. Let F, $G \in S_k(N,\chi)$ be Siegel cusp forms of level N and let M be a natural number which divides N. For each $\gamma \in \mathrm{Sp}_2(\mathbf{Z})$, we write

$$F|_k\gamma(Z) = \sum_{n\geq 1} \phi_{n,\gamma}(\tau,z)\mathbf{e}\left(\frac{n\tau'}{N}\right),$$
$$G|_k\gamma(Z) = \sum_{n\geq 1} \psi_{n,\gamma}(\tau,z)\mathbf{e}\left(\frac{n\tau'}{N}\right).$$

Then we define the *Rankin convolution series* $D_{F,G;M}(s)$ as $\zeta(2s-2k+4)$ times

$$(1)\quad \sum_{n\geq 1}\left\{\int_{\mathcal{F}} \sum_{\gamma\in\Gamma_2(N)\backslash\Gamma_2(M)} \phi_{n,\gamma}(\tau,z)\,\bar{\psi}_{n,\gamma}(\tau,z) \times \exp\left(-\frac{4\pi n y^2}{vN}\right) v^{k-3}\,du\,dv\,dx\,dy\right\} n^{-s},$$

where $\mathcal{F}$ is a fundamental domain $\Gamma_1^J(M)\backslash\mathcal{H}_1\times\mathbf{C}$, and define its gamma factor by

$$D^*_{F,G;M}(s) := (2\pi)^{-2s}\,\Gamma(s)\,\Gamma(s-k+2)\,D_{F,G;M}(s).$$

In a special case of $M = N$, this is an obvious generalization of the symmetric square series defined by Rankin in the case of genus 1 ([10]):

$$D_{F,G;N}(s) = \frac{1}{N^s}\zeta(2s-2k+4)\sum_{n\geq 1}\frac{\langle\phi_n,\psi_n\rangle_N}{n^s},$$

where ϕ_n (resp. ψ_n) denotes the n-th Fourier-Jacobi coefficient of F (resp. G).

On the other hand, if $F(Z) \in S_k(N,\chi)$ is a Hecke eigenform with $T(n)F = \lambda_F(n)F$ for all the Hecke operators $T(n)$ with $(n,N) = 1$, one can associate with F the *spinor zeta function* which is an Euler product of the form

$$(2)\quad Z_F(s) := \prod_{\substack{p:\mathrm{prime}\\(p,N)=1}} Q_{F,p}(p^{-s})^{-1},$$

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