Milnor numbers and classes of local complete intersections

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1. Introduction. Let V be an n-dimensional compact complex subvariety of a complex manifold M. When V is non-singular, the Chern classes of the complex tangent bundle TV are well-defined cohomology classes in $H^*(V;\mathbf{Z})$. We denote by $c_*(V)$ their image by the Poincaré isomorphism

$$P_V : H^{2(n-i)}(V;\mathbf{Z}) \xrightarrow{\frown [V]} H_{2i}(V;\mathbf{Z}),$$

cap-product by the fundamental class $[V]$ of V. When V is singular there is no more Chern cohomology classes, but there are several theories generalizing homology classes $c_*(V)$. For instance, the Chern-Schwartz-MacPherson classes $c_*^{SM}(V)$ ([16], [17], [10], [3]) and the Fulton-Johnson classes $c_*^{FJ}(V)$ [5] are two different theories which coincide with $c_*(V)$ when V is non-singular. Our main purpose is to compare the Chern-Schwartz-MacPherson and the Fulton-Johnson classes when V is a local complete intersection. In this paper, we give a presentation of the main results; the complete proofs will be published elsewhere (see [4]).

On one hand, M. H. Schwartz defined actually classes in $H^*(M, M-V;\mathbf{Z})$ ([16], 1965). Let us denote by m the complex dimension of M. It is proved in [3](1979) that Schwartz classes are mapped by the Alexander duality

$$H^{2(m-i)}(M, M-V;\mathbf{Z}) \longrightarrow H_{2i}(V;\mathbf{Z})$$

onto the classes defined by MacPherson ([10], 1974).

We restrict ourselves to the case of a local complete intersection V defined by a holomorphic section of a vector bundle. We consider a holomorphic vector bundle $E \to M$ of rank $k = m - n$, and a holomorphic section s generically transverse to the zero section, such that V is the zero set $s^{-1}(0)$. In this case, the virtual classes of V are defined in [4] as the Chern classes $c_{vir}^*(V) \in H^*(V;\mathbf{Z})$ of the "virtual tangent bundle" $[TM - E]|_V$ (in the complex K-theory $\tilde{K}(V)$). The virtual classes $c_{vir}^*(V)$ coincide with the usual Chern classes if V is non-singular and their images by the Poincaré duality (no more an isomorphism), denoted by $c_*^{vir}(V)$, coincide with the Fulton-Johnson classes $c_*^{FJ}(V)$.

In order to compare the Schwartz-MacPherson and the Fulton-Johnson classes of a local complete intersection, we have to study the difference $c_*^{vir}(V) - c_*^{SM}(V)$. This difference localizes near the singular part $\mathrm{Sing}(V)$ of V: more precisely, if we denote by $(S_\alpha)_\alpha$ the family of connected components of $\mathrm{Sing}(V)$, there are well defined elements $\mu_*(V, S_\alpha)$ in $H_*(S_\alpha;\mathbf{Z})$, called "the (homological) Milnor classes" of V at S_α, such that we get the

Theorem A. *We have,*

$$c_*^{vir}(V) - c_*^{SM}(V) = (-1)^n \sum_\alpha (i_\alpha)_*\big(\mu_*(V, S_\alpha)\big),$$

where $(i_\alpha)_* : H_*(S_\alpha) \to H_*(V)$ denotes the natural map arising from the inclusion $S_\alpha \subset V$.

The Milnor number is well defined by Milnor [11], for hypersurfaces with isolated singular points, by Hamm [7] and Lê [8] for local complete intersections still with isolated singular points, and by Parusiński [12] for hypersurfaces with any compact singular set. The following theorem justifies the terminology "Milnor class" that we use.

Theorem B. *$\mu_0(V, S_\alpha)$ is equal to the Milnor number of V at S_α in $H_0(S_\alpha) \cong \mathbf{Z}$, in all situations where this number has been already defined.*

Such a theory for Milnor classes in homology has also been suggested by Yokura [21], and given in the case of complex compact hypersurfaces by Aluffi [1] and Parusiński-Pragacz [14].

For $r \geq 1$, we explain how to compute the Milnor class $\mu_{r-1}(V, S_\alpha)$ by means of an r-frame $F^{(r)}$ defined on the regular part V_0 of V near (but off) S_α, as the difference (up to sign) of two classes of $F^{(r)}$ at S_α, the so-called "Schwartz class" and the "virtual class" (Theorems C and D).

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