On semicontinuous solutions for general Hamilton-Jacobi equations

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1. Introduction. This is an announcement of our recent work [16], where detailed proofs are given as well as extensions. We do not give the proof of statements. We consider the initial value problem for the Hamilton-Jacobi equation of form

(1a) $u_t + H(x, u_x) = 0 \quad \text{in } (0,T) \times \mathbf{R}^n,$

(1b) $u(0,x) = u_0(x), \qquad x \in \mathbf{R}^n,$

where $u_t = \partial u/\partial t$ and $u_x = (\partial_{x_1} u, \dots, \partial_{x_n} u)$, $\partial_{x_i} u = \partial u/\partial x_i$; $\infty \geq T > 0$ is a fixed number. Our main goal is to find a suitable notion of solution when u_0 is discontinuous. The theory of viscosity solutions initiated by Crandall and Lions [6] yields the global solvability of the initial value problem by extending the notion of solutions when u_0 is continuous (cf. [8, Chap.10], [15], [2]). In fact, if initial data u_0 is bounded, uniformly continuous, it is well-known [6], [15] that the initial value problem (1a)-(1b) admits a unique global (uniformly) continuous viscosity solution when H is enough regular, for example H satisfies the Lipschitz conditions

(2a) $|H(x,p) - H(x,q)| \leq C|p-q|$

(2b) $|H(x,p) - H(y,p)| \leq C(1+|p|)|x-y|.$

We only refer to [2], [15] and [7] for the basic theory of viscosity solutions. The notion of viscosity solution has been extended to semicontinuous functions. This is very important to prove the existence of solutions without appealing hard estimates. Such a method is first introduced by [13]. However, if u_0 is, for example, upper semicontinuous, a classical semicontinuous viscosity solution may not be unique.

Recently to overcome this inconvenience, Barron and Jensen [3] introduced another notion of viscosity solutions for semicontinuous functions when the Hamiltonian $H = H(x,p)$ is concave in p and proved the existence and the uniqueness of their solution for (1a), (1b) for bounded (from above), upper semicontinuous initial data u_0. Their solution is now called a bilateral viscosity solution [1]. For later development of the theory as well as other approaches we refer to [1] and references cited there. However, their theory is limited for concave H. (In [3] H is assumed to be convex but they consider the terminal value problem which is easily transformed to the initial value problem with concave Hamiltonian by setting $T-t$ by t.)

In this paper we introduce a new notion of a solution which is unique for a given initial upper semicontinuous initial data. For (1a), (1b) we consider auxiliary problem

(3a) $\psi_t - \psi_y H(x, -\psi_x/\psi_y) = 0 \quad \text{in } (0,T) \times \mathbf{R}^{n+1},$

(3b) $\psi(0,x,y) = \psi_0(x,y), \qquad (x,y) \in \mathbf{R}^n \times \mathbf{R}.$

The equation (3a) is called the level set equation for the evolution of the graph of u of (1a). In fact, if a level set of a solution ψ of (3a) is given as the graph of a function $v = v(t,x)$, then v must solve (1a). For given upper semicontinuous initial data $u_0 : \mathbf{R}^n \to \mathbf{R} \cup \{-\infty, +\infty\}$, shortly $u_0 \in USC(\mathbf{R}^n)$, we take

(4) $\psi_0(x,y) = -\min\{\mathrm{dist}((x,y), K_0),\ 1\},$

where

(5) $K_0 = \{(x,y) \in \mathbf{R}^n \times \mathbf{R};\ y \leq u_0(x)\}.$

We solve (3a), (3b) and set

(6) $\overline{u}(t,x) = \sup\{y \in \mathbf{R};\ \psi(t,x,y) \geq 0\},$

where ψ is the continuous viscosity solution of (3a), (3b). We call $\overline{u}$ an *L-solution* of (1a), (1b). Such a solution uniquely exists globally in time under suitable condition on H.

Theorem 1. *Assume that the recession function*

(7) $H_\infty(x,p) = \lim_{\lambda \downarrow 0} \lambda H(x, p/\lambda), \quad x \in \mathbf{R}^n,\ p \in \mathbf{R}^n$

exists and that H satisfies (2a), (2b). *Then there exists a global unique L-solution for an arbitrary*

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