Dynamics of composite functions

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Abstract: Let f and g be two transcendental entire functions. In this paper, mainly by using Iversen's theorem on the singularities, we studied the dynamics of composite functions. We have proved that the Fatou sets of $f \circ g$ and $g \circ f$ have the same dynamical properties.

Key words: Entire function; complex dynamics; composite functions.

1. Introduction. Let $f(z)$ be a nonlinear entire function. The sequence of the iterates of f is denoted by

$$f^{n+1} = f^n \circ f$$

where $f^0 = id, f^1 = f$. We define $F = F(f)$ to be the largest open set in which the iterates of f form a normal family, and

$$J = J(f) = C - F(f).$$

They are called the Fatou set and Julia set of f, respectively.

Suppose U is a component of the Fatou set of f, U is called a wandering domain if $f^m(U) \cap f^n(U) = \phi$ for $m \neq n$. If U is not wandering, we call U a pre-periodic component of f. That is, $f^n(f^m(U)) = f^m(U)$ for $n, m \geq 0$. If $m = 0$, we call U a periodic component of f. D. Sullivan, see, e.g. [9] proved that the Fatou set of any rational function has no wandering domain; I. N. Baker and others, see, e.g. [2] gave examples to show that transcendental entire functions may have wandering domains. In [1], it is known that functions which have only a finite number of asymptotic and critical values have no wandering domain. I. N. Baker and A. P. Singh [3] in 1995 proved that if $p(z)$ is a non-constant entire function and $g(z) = a + be^{2\pi iz/c}$, where a, b and c are non-zero constants, such that $g \circ p$ has no wandering domain, then so does $p \circ g$. We have generalized this and proved that if f and g are two given transcendental entire functions, then $f \circ g$ has wandering domains if and only if $g \circ f$ does. Moreover, we have shown that the dynamics of $f \circ g$ and $g \circ f$ are very similar.

2. The lemmas and main results.

Lemma 2.1. (Iversen's theorem, see [7]) *Let F be a Riemann surface of parabolic type over the w plane, and let $w = w_0$ be an arbitrary point in the plane. Further assume that $\delta > 0$ and that w_1 is an interior point of the surface F with $|w_1 - w_0| = \delta$. Then it is possible to find a continuous curve L that joins the points w_1 and w_0 without leaving the disk $|w - w_0| < \delta$ and that with the possible exception of the end point w_0 consists of nothing but interior points of the surface F.*

Concerning the components of the Fatou set, we have the following two lemmas:

Lemma 2.2 (I. N. Baker [1]). *Let f be a transcendental entire function. Then every unbounded component U of $F(f)$ is simply connected.*

Lemma 2.3 (I. N. Baker [1]). *Let f be a transcendental entire function. Then any pre-periodic Fatou component U is simply connected, and therefore any multiple-connected Fatou component is bounded and wandering.*

Theorem 2.1. *Suppose that f and g are entire functions. Then $g \circ f$ has no wandering domain, if and only if $f \circ g$ has no wandering domain.*

Theorem 2.2. *Suppose that f and g both are transcendental entire functions. Then $f \circ g$ contains a Schröder domain if and only if $g \circ f$ does. In addition, similar conclusions hold for a Leau domain, Siegel disc, Baker domain and Böttcher domain.*

3. Proofs of Theorems.

3.1. Proof of Theorem 2.1.

Proof. We first assume that $g \circ f$ has no wandering domain. Let $K = f \circ g$ and $H = g \circ f$. Then we have $H \circ g = g \circ K$. Suppose on the contrary that K has a sequence of wandering do-

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