A Yang-Mills-Higgs gradient flow on $\boldsymbol{R}^3$ blowing up at infinity

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1. Yang-Mills-Higgs functional. We prove long time existence of the Yang-Mills-Higgs gradient flow on Euclidean 3-space $\boldsymbol{R}^3$, with a geometric characterization at the singular points. Since a solution of the Yang-Mills-Higgs gradient flow constructed in this paper has geometrically reasonable properties at the ideal boundary of $\boldsymbol{R}^3$, we are motivated to propose our definition of a global solution for the gradient flow.

Let P be the trivial bundle $\boldsymbol{R}^3 \times SU(2)$ over $\boldsymbol{R}^3$ and let $\tilde{C}$ be the set of pairs of connections A on the principal bundle P and Higgs field Φ on $\boldsymbol{R}^3$; an $\mathfrak{su}(2)$-valued map on $\boldsymbol{R}^3$, where $\mathfrak{su}(2)$ is the Lie algebra of $SU(2)$. The Yang-Mills-Higgs functional is a functional on C defined by the following: for $(A, \Phi) \in \tilde{C}$,

$$E(A, \Phi) = \int_{R^3} (|F_A|^2 + |d_A \Phi|^2) dV \tag{1}$$

where d_A is the covariant exterior differentiation on the bundle P and F_A denotes the curvature 2-form of A. Critical points of the functional (1) are called *Yang-Mills-Higgs configurations.*

2. Yang-Mills-Higgs gradient flow. We define the following compactified configuration space (cf. Groissor [2]):

$$C = \{(A, \Phi) : E(A, \Phi) < \infty, \quad |\Phi(x)| \to 1 \text{ as } |x| \to \infty\}.$$

The configuration space C has a geometric invariant, $N(A, \Phi)$, defined by

$$N(A, \Phi) = \frac{1}{4\pi} \int_{R^3} F_A \wedge d_A \Phi. \tag{2}$$

$N(A, \Phi)$ is called the *monopole number* (or *magnetic charge*) of (A, Φ). Groissor [2] showed that if $(A, \Phi) \in C$, then $N(A, \Phi)$ is an integer and the functional $N : C \to \boldsymbol{Z}$ gives a path component decomposition on C. Restricting Φ to a sufficiently large 2-shpere S^2 in $\boldsymbol{R}^3$ determines a homotopy class of maps on S^2. Let S_∞ be the ideal boundary of $\boldsymbol{R}^3$. We can identify S_∞ with the unit 2-sphere $S^2(1)$ canonically: given Φ, we define a map $\hat{\Phi} : S_\infty \to S^2$ by

$$\hat{\Phi}(\omega) = \lim_{r\to\infty} \frac{\Phi(r, \omega)}{|\Phi(r, \omega)|}, \tag{3}$$

where $\Phi(r, \omega) = \Phi(x)$, $r = |x|$, $\omega = \frac{x}{|x|}$. Then, we have $N(A, \Phi) = -\deg(\hat{\Phi})$. Furthermore, $2N(A, \Phi)$ gives the first Chern number of some bundle over S^2. Thus, in constructing a solution of (4), it is reasonable to take its behavior at the ideal bundary S_∞ into account.

We consider the following heat flow associated with the Yang-Mills-Higgs functional (1):

$$\begin{cases} \partial_t A = - d_A^* F_A - [\Phi, d_A \Phi], \\ \partial_t \Phi = \Delta_A \Phi, \end{cases} \tag{4}$$

with the initial condition $(A(0), \Phi(0)) = (A_0, \Phi_0)$.

We call a curve $(A(t), \Phi(t))$ in the configuration space C a *smooth solution* of (4) if $(A(t), \Phi(t))$ satisfies (4) in the classical sense. To fix the geometrical meaning for solutions of (4), we introduce the following notion:

Definition 1. A smooth solution $(A(t), \Phi(t))$ of (4) is called *extendable* on $(0, T]$ if the following conditions are satisfied:

(i) For each $t \in (0, T]$, there exists a gauge transformation $g(t)$ such that $g^*(t)A(t)$ extends to a smooth connection over $S_\infty \cong S^2(1)$.

(ii) $N(A(t), \Phi(t))$ of (4) is independent of $t \in (0, T]$.

Let ε be a positive constant. For $\omega_0 \in S^2(1)$, let $B_\tau(\omega_0)$ be the geodesic ball centered at ω_0 with the radius τ.

Definition 2. A smooth solution $(A(t), \Phi(t))$ of (4) of has the *ε-property* if

$$\liminf_{r\to\infty} \int_{B_r(\omega_0)} r^2(|F_A(t, r, \omega)| + |d_A \Phi(t, r, \omega)|) d\omega \leq \varepsilon, \tag{5}$$

for sufficiently small τ, for all $t \in (0, T]$ and for all $\omega_0 \in S^2$.

This definition gives a criterion for obtaining an extendable solution, and is one of the fundamental observations for constructing a global

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