Power series and asymptotic series associated with the Lerch zeta-function

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1. Introduction. Let s be a complex variable, α and λ real parameters with $\alpha > 0$. Let $\Gamma(s)$ be the gamma-function and $(s)_n = \Gamma(s+n)/\Gamma(s)$ for any integer n denote Pochhammer's symbol. The zeta-function

$$(1.1)\quad \phi(\lambda, \alpha, s) = \sum_{n=0}^{\infty} e^{2\pi i n\lambda}(n+\alpha)^{-s} \quad (\mathrm{Re}\, s > 1)$$

was first introduced and studied by Lerch [10] and Lipschitz [11]. For $\lambda \in \boldsymbol{R}\setminus\boldsymbol{Z}$ it is continued to an entire function over the s-plane, while if $\lambda \in \boldsymbol{Z}$ it reduces to the Hurwitz zeta-function $\zeta(s, \alpha)$. Note that $\zeta(s, 1) = \zeta(s)$ is the Riemann zeta-function.

It is the main aim of the present paper to study power series and asymptotic series for the Lerch zeta-function $\phi(\lambda, \alpha, s)$ in the second parameter (see (1.6) and (2.2) below), based on Mellin-Barnes type of integral formulae. Two applications of our main result will also be presented. For that purpose we extend the domain of the second parameter as follows. Let ω be a real number fixed arbitrarily with $-\pi/2 < \omega < \pi/2$, and S_ω denote the sectorial domain $-\pi/2 + \omega < \arg z < \pi/2 + \omega$. First for any parameter z in S_0, the analytic continuation of $\phi(\lambda, z, s)$ over the s-plane is given by the formula

$$(1.2)\quad \phi(\lambda, z, s) = \frac{1}{\Gamma(s)(e^{2\pi i s-1})}\int_{\mathscr{C}} \frac{e^{-zw}w^{s-1}}{1-e^{2\pi i\lambda-w}}\,dw,$$

where $\mathscr{C}$ is the contour which starts from infinity, proceeds along the real axis to a small positive δ, rounds the origin counter-clockwise, and returns to infinity along the real axis; $\arg w$ varies from 0 to 2π round $\mathscr{C}$. The expression on the right-hand side of (1.2) shows that $\phi(\lambda, z, s)$ is also an analytic function of z in S_0. Next if z is in the intersection of S_0 and S_ω, then the contour $\mathscr{C}$ can be rotated around the origin by an angle $-\omega$ without altering the value of the integral. The resulting formula provides the analytic continuation of $\phi(\lambda, z, s)$ over the s-plane, for any z in S_ω. This operation shows that the domain of z in $\phi(\lambda, z, s)$ can be extended to the whole sector $|\arg z| < \pi$. When $0 < \lambda \leq 1$ and $0 < \alpha \leq 1$, it follows from (1.2) the functional equation (cf. Erdélyi *et al.* [6], p. 26 and p. 29])

$$(1.3)\quad \phi(\lambda, \alpha, s) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}}\Big\{e^{\frac{1}{2}\pi i(1-s)}\sum_{l=0}^{\infty} e^{-2\pi i\alpha(l+\lambda)}(l+\lambda)^{s-1} + e^{\frac{1}{2}\pi i(s-1)}\sum_{l=0}^{\infty}{}' e^{2\pi i\alpha(l+1-\lambda)}(l+1-\lambda)^{s-1}\Big\} \quad (\mathrm{Re}\, s < 0),$$

where the prime on the latter summation symbol indicates that the term corresponding to $l = 0$ is to be omitted if $\lambda = 1$.

The present investigation was motivated by the following results (1.4), (1.5) and (1.6), for which we give a brief overview before starting our discussion. It is classically known that the asymptotic expansion

$$(1.4)\quad \sum_{n=1}^{N} n^{-s} \sim \zeta(s) - \frac{N^{1-s}}{s-1} + \frac{1}{2}N^{-s} - \sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}(s)_{2k-1}N^{-s-2k+1},$$

as $N \to +\infty$, holds for all $s \neq 1$, where B_k $(k \geq 0)$ is the k-th Bernoulli number. Berndt [4, p. 150] attributed this formula to Ramanujan. Next let $\Psi(a, c; z)$ denote the solution of Kummer's confluent hypergeometric differential equation $zu'' + (c-z)u' - au = 0$ satisfying $\Psi(a, c; z)$

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