On the Z_3-extension of a certain cubic cyclic field

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In our previous paper [2], we gave the following Theorem for vanishing of Iwasawa invariants of a cyclic extension of odd prime degree over the rational number field $\boldsymbol{Q}$.

Theorem A ([2, Cor. 3.6.]). *Let l be an odd prime number, k a cyclic extension of degree l over $\boldsymbol{Q}$, $\boldsymbol{Q}_\infty$ the cyclotomic $\boldsymbol{Z}_l$-extension of $\boldsymbol{Q}$ and $k_\infty = k\boldsymbol{Q}_\infty$ the composite field of k and $\boldsymbol{Q}_\infty$. Then the following are equivalent*:

(1) *The Iwasawa λ-invariant $\lambda_l(k_\infty/k)$ of k_∞ over k is zero.*

(2) *For any prime ideal $\mathfrak{p}$ of k_∞ which is prime to l and ramified in k_∞ over $\boldsymbol{Q}_\infty$, the order of the ideal class of $\mathfrak{p}$ is prime to l.*

Moreover, using Theorem A, we gave some examples of vanishing of $\lambda(k_\infty/k)$, in [2]. More precisely, let $\boldsymbol{Q}_1$ be the initial layer of the cyclotomic $\boldsymbol{Z}_3$-extension $\boldsymbol{Q}_\infty$ of $\boldsymbol{Q}$, k a cubic cyclic extension over $\boldsymbol{Q}$ with prime conductor p such that $p \equiv 1 \pmod 9$, $k_1 = k\boldsymbol{Q}_1$, $E_{\boldsymbol{Q}_1}$ (resp. E_{k_1}) the unit group of $\boldsymbol{Q}_1$ (resp. k_1) and $N_{k_1/\boldsymbol{Q}_1}$ the norm k_1 over $\boldsymbol{Q}_1$. In [2, Example 4.1], we treated the case $(E_{\boldsymbol{Q}_1} : N_{k_1/\boldsymbol{Q}_1}(E_{k_1})) = 9$ and $p \not\equiv 1 \pmod{27}$, which implies that the prime ideals of k_1 lying above p are principal by genus formula. In this paper, we treat the case $p = 73$, which could not be treated in [2]. We note that if $p = 73$, then $(E_{\boldsymbol{Q}_1} : N_{k_1/\boldsymbol{Q}_1}(E_{k_1})) = 3$ (cf. [2, Example 4.2]).

The main purpose of this paper is to prove the following theorem:

Theorem. *Let $\zeta_{73} = e^{\frac{2\pi i}{73}}$, k the unique subfield of $\boldsymbol{Q}(\zeta_{73})$ of degree 3 over $\boldsymbol{Q}$ and k_∞ the cyclotomic $\boldsymbol{Z}_3$-extension of k. Then the λ-invariant $\lambda_3(k_\infty/k)$ of k_∞ over k is zero.*

The Theorem will be proved by using Fukuda's method (cf. [1]). We note that Leopoldt's conjecture is valid for the above k (cf. [4, p. 71]) and k is totally real. Now we explain notations.

We denote by $\boldsymbol{Z}$ the rational integer ring. We put $\zeta_n = e^{\frac{2\pi i}{n}}$ for a positive integer n. Let F be a number field. We denote by O_F the integer ring of F. For an integral ideal $\mathfrak{a}$ of F, we denote by $Cl(\mathfrak{a})$ the ideal class of $\mathfrak{a}$, $O_F/\mathfrak{a}$ the factor ring of O_F over $\mathfrak{a}$ and $(O_F/\mathfrak{a})^\times$ the set of invertible elements of $O_F/\mathfrak{a}$. For a Galois extension L of F, we denote by $G(L/F)$ the Galois group of L over F. Let G be a group. For elements $g_1, g_2, \ldots, g_r$ of G, we denote by $\langle g_1, g_2, \ldots, g_r \rangle$ the subgroup of G generated by $g_1, g_2, \ldots, g_r$.

In order to prove our Theorem, we shall use the following Lemma:

Lemma 1 (cf. [3, Cor. of Prop. 1]). *Let F be a totally real number field for which Leopoldt's conjecture is valid. Let A_0 be the l-sylow subgroup of the ideal class group of F and $\mathfrak{a}$ a product of primes of F lying above l such that $Cl(\mathfrak{a}) \in A_0$. Then $\mathfrak{a}$ becomes principal in the n-th layer F_n of F_∞ over F for sufficiently large n.*

Let $\boldsymbol{Q}_\infty$ be the cyclotomic $\boldsymbol{Z}_3$-extension of $\boldsymbol{Q}$ and $\boldsymbol{Q}_n$ the n-th layer of $\boldsymbol{Q}_\infty$ over $\boldsymbol{Q}$ for a non-negative integer n. We let $k_n = k\boldsymbol{Q}_n$ and A_n the 3-sylow subgroup of the ideal class group of k_n. We put $\theta = \zeta_9 + \zeta_9^{-1} = 2\cos\frac{2\pi}{9}$. Then the roots of the equation $x^3 - 3x + 1 = 0$ are θ, $\theta^2 - 2 = \zeta_9^7 + \zeta_9^{-7}$ and $-\theta^2 - \theta + 2 = \zeta_9^4 + \zeta_9^{-4}$. We note $\boldsymbol{Q}_1 = \boldsymbol{Q}(\theta)$ and $x^3 - 3x + 1 \equiv (x+34)(x+14)(x+25) \pmod{73}$. Let $\mathfrak{p}_1$ be the ideal $(\theta + 34, 73)$ of $O_{\boldsymbol{Q}_1}$ generated by $\theta + 34$, 73. Since $N_{\boldsymbol{Q}_1/\boldsymbol{Q}}(\theta^2 + 6\theta - 3) = (\theta^2 + 6\theta - 3)(5\theta^2 - \theta - 11)(-6\theta^2 - 5\theta + 11) = -73$ and since $\theta^2 + 6\theta - 3 \equiv (\theta + 34)(\theta - 28) \pmod{73}$, we have $\mathfrak{p}_1 = (\theta^2 + 6\theta - 3)$. In a similar way, we have $(\theta + 14, 73) = (5\theta^2 - \theta - 11)$ and $(\theta + 25, 73) = (-6\theta^2 - 5\theta + 11)$. We put $\mathfrak{p}_2 = (5\theta^2 - \theta - 11)$ and $\mathfrak{p}_3 = (-6\theta^2 - 5\theta + 11)$. Note that $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3$ are the distinct prime ideals of $\boldsymbol{Q}_1$ lying above 73 and $(O_{\boldsymbol{Q}_1}/\mathfrak{p}_i)^\times \cong (\boldsymbol{Z}/73\boldsymbol{Z})^\times$.

We put $P\mathfrak{m} = \{a \in \boldsymbol{Q}_1 ; a \text{ is prime to } \mathfrak{m}\}$ and $S\mathfrak{m} = \{a \in P\mathfrak{m} ; a \equiv 1 \pmod{\mathfrak{m}}\}$ for an ideal $\mathfrak{m}$ of $\boldsymbol{Q}_1$. Now, we define a surjective homomorphism φ of P_{73}/S_{73} to an abelian group $V =$