On Bougerol and Dufresne's identities for exponential Brownian functionals

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1. Introduction. Let $B = \{B_t\}_{t \geqq 0}$ be a one-dimensional standard Brownian motion starting from 0. To $\{X_t^{(\nu)} = B_t + \nu t\}_{t \geqq 0}$, a Brownian motion with constant drift ν, we associate the exponential additive functional

$$A_t^{(\nu)} = A_t^{(\nu)}(B) = \int_0^t \exp(2(B_s + \nu s))ds, \ t \geqq 0.$$

This Wiener functional plays an important role in a number of domains, mathematical finance (Yor [13], Leblanc [11]), diffusion processes in random environments (Comtet-Monthus [4], Comtet-Monthus-Yor [5], Kawazu-Tanaka [10]), probabilistic studies related to Laplacians on hyperbolic spaces (Gruet [8], Ikeda-Matsumoto [9]) and so on. The readers can find more related topics and references in [16].

We denote A_t for $A_t^{(0)}$. The joint law of (A_t, B_t) is fairly complicated (Yor [13]), although it is quite tractable. But Bougerol's identity ([3]),

$$\text{(1.1)} \quad \sinh(B_t) \overset{\text{(law)}}{=} \gamma_{A_t} \text{ for any fixed } t > 0$$

for another Brownian motion $\{\gamma_s\}_{s \geqq 0}$ independent of B, makes it easy to calculate the Mellin transform of the probability law of A_t. A simple proof of (1.1), as a consequence of Itô's formula, has been provided in Alili-Dufresne-Yor [1] who have shown the identity in law of the processes

$$\{\exp(B_t)\int_0^t \exp(-B_s)d\gamma_s\}_{t \geqq 0} \text{ and } \{\sinh(B_t)\}_{t \geqq 0}.$$

Another approach to the joint probability law of (A_t, B_t) is found in Alili-Gruet [2] and Ikeda-Matsumoto [9]. These authors have shown independently the following formula. Let ϕ be a function defined by

$$\text{(1.2)} \quad \phi(x, z) = \sqrt{2}\, e^{x/2}(\cosh z - \cosh x)^{1/2}, \ z \geqq |x|.$$

Then it holds that

$$\text{(1.3)} \quad E[\exp(-u^2A_t/2) \,|\, B_t = x]\frac{1}{\sqrt{2\pi t}}\exp(-x^2/2t) = \int_{|x|}^\infty \frac{z}{\sqrt{2\pi t^3}}\exp(-z^2/2t)J_0(u\phi(x, z))dz,$$

for every $t > 0$, where $E[\,|\,]$ is the conditional expectation with respect to the Wiener measure and J_0 is the Bessel function of the first kind of order 0.

Moreover recall that $J_0(\xi)$ is the characteristic function of a symmetrized arcsine random variable $2Z - 1$, where Z is a usual arcsine variable whose probability density is $(\pi \cdot \sqrt{z(1-z)})^{-1}$, $0 < z < 1$. Then, following [2], we can show

$$\text{(1.4)} \quad (\gamma_{A_t}, B_t) \overset{\text{(law)}}{=} ((2Z-1)\phi(B_t, \sqrt{R_t^2 + B_t^2}), B_t)$$

as a consequence of (1.3), where $R = \{R_t\}_{t \geqq 0}$ is a two-dimensional Bessel process starting from 0 independent of B and Z. We will also see below that (1.4) is equivalent to

$$(\gamma_{A_t}, B_t) \overset{\text{(law)}}{=} ((2Z-1)\phi(B_t, |B_t| + L_t), B_t)$$

for the local time L_t of B at 0. See Lemmas 1 and 2 in Section 3 for details.

The origin of the present note is the following integral moments formula which has been obtained by Ikeda-Matsumoto [9] by using a result in Yor [13]: for every positive integer n and every $x \in \mathbf{R}$, it holds that

$$\text{(1.5)} \quad E[(A_t)^n | B_t = x]\frac{1}{\sqrt{2\pi t}}\exp(-x^2/2t) = \frac{\exp(nx)}{\sqrt{2\pi t^3}\,n!}\int_{|x|}^\infty b\exp(-b^2/2t)(\cosh b - \cosh x)^n db.$$

Recall that, letting $\mathbf{e}$ be a standard exponential random variable whose density function is $\exp(-x)$, $x \geqq 0$, we have $E[\mathbf{e}^n] = n!$ and

$$P(\sqrt{2\mathbf{e}t} > b | \sqrt{2\mathbf{e}t} > |a|) = \exp(-a^2/2t)\int_b^\infty \frac{c}{t}\exp(-c^2/2t)dc, \ b > |a|.$$

Then (1.5) is equivalent to

$$\text{(1.6)} \quad E[(e^{-a}\mathbf{e}A_t)^n | B_t = a] = E[(\cosh\sqrt{2\mathbf{e}t} - \cosh a)^n | \sqrt{2\mathbf{e}t} > |a|],$$

where $\mathbf{e}$ is assumed to be independent of B. Although Carleman's sufficient condition for the

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