Dynamics of composite mappings

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Abstract: In this paper, we will prove some theorems that relate to the dynamics of a composite mapping and its two factors.

Key words: Dynamics; Fatou-Julia theory; composite mappings.

1. Introduction. In this short note, we will generalize some theorems of complex dynamics of one variable for composite functions to several variables cases. In particular, we will prove the following result:

Main theorem. Let f and g be holomorphic self-mappings of degree ≥ 2 on the complex projective space $\boldsymbol{P}^m$ of dimension m. If f, g satisfy $f \circ g = g \circ f$, then $J_{equ}(f) = J_{equ}(g)$.

Here $J_{equ}(f)$ is the Julia set of the mapping f. For the rational function case, that is, $m = 1$, the main theorem gives Theorem 4.2.9 of [3], due to Beardon. For more information on this topic, see, e.g., [1] and [4]. The proof of the main theorem is based on the method used by Beardon and uses a result obtained recently by Ueda [10].

2. Proof of the main theorem. Given a metric space (M, d), denote the set of continuous self-mappings on M by $C(M, M)$. Fix $f \in C(M, M)$. Then there is a maximal open subset so called the Fatou set $F_{equ}(f) = F_{equ}(f, d)$ of M on which the family of iterates $\{f^n\}$ is equicontinuous. Define the Julia set

$$J_{equ}(f) = J_{equ}(f, d) = M - F_{equ}(f, d).$$

It is easy to prove that the sets $F_{equ}(f)$ and $J_{equ}(f)$ are backward invariant if f is an open mapping. Some basic properties of sets $F_{equ}(f)$ and $J_{equ}(f)$ are discussed in Hu-Yang [6] and [7]. By adopting the argument used by Beardon in his proof of Theorem 4.2.9 in [3], the following general result can be obtained:

Theorem 2.1. *If $f, g \in C(M, M)$ are open with $f \circ g = g \circ f$ and satisfy some Lipschitz condition*

$$d(f(x), f(y)) \leq \lambda d(x, y), \; d(g(x), g(y)) \leq \lambda d(x, y),$$

on M, then $f^n(F_{equ}(g)) \subset F_{equ}(g)$ and $g^n(F_{equ}(f)) \subset F_{equ}(f)$ for all $n \in \boldsymbol{Z}_+$.

Proof. For any set E, we denote the diameter of E by diam $[E]$ computed using the metric d. Now take $x \in F_{equ}(f)$. By the equicontinuity of $\{f^n\}$ at x, given any positive ε, there is a positive δ such that for all n,

$$\mathrm{diam}[f^n(M(x;\delta))] < \varepsilon/\lambda.$$

As f and g commute we deduce that

$$\begin{aligned} &\mathrm{diam}[f^n \circ g(M(x;\delta))] \\ &\quad = \mathrm{diam}[g \circ f^n(M(x;\delta))] \\ &\quad \leq \lambda \mathrm{diam}[f^n(M(x;\delta))] < \varepsilon. \end{aligned}$$

It follows that $\{f^n\}$ is equicontinuous at $g(x)$, so, in particular, $g(x) \in F_{equ}(f)$. This proves that g, and hence each g^n, maps $F_{equ}(f)$ into itself. We conclude that $g^n : F_{equ}(f) \to F_{equ}(f)$, and so, by symmetry, $f^n : F_{equ}(g) \to F_{equ}(g)$. □

Note that the rational function case is contained implicitly in the proof of Theorem 4.2.9 of [3]. For more information on this topic, see, e.g., [1] and [4]. As an extension of Beardon's result, we prove the following:

Corollary 2.1. *If $f, g \in \mathcal{H}_d$ with $d \geq 2$ satisfying $f \circ g = g \circ f$, then $J(f) = J(g)$, where $\mathcal{H}_d$ is the space of the holomorphic self-mappings on $\boldsymbol{P}^m$ given by homogeneous polynomials of degree d.*

We first note that any C^1 mapping f of a compact Riemannian manifold M satisfies some Lipschitz condition

$$d_M(f(x), f(y)) \leq \lambda d_M(x, y),$$

where d_M is the distance function induced by the

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