On smooth projective threefolds with non-trivial surjective endomorphisms

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The main purpose of this paper is to announce the structure of non-singular projective threefold X with a surjective morphism $f: X \to X$ onto itself, which is not an isomorphism.
We call it a non-trivial surjective endomorphism of X.
The details will be published elsewhere.

Lemma 1. Let $f: X \to X$ be a surjective morphism from a non-singular projective variety X onto itself. Then
(1) f is a finite morphism.
(2) If $\kappa(X) \geq 0$, f is a finite étale morphism.
(3) If f is a finite étale morphism, then $\chi(\mathcal{O}_X) = \deg(f) \cdot \chi(\mathcal{O}_X)$.

The structure of algebraic surfaces with a non-negative Kodaira dimension, which admit a non-trivial surjective endomorphism, are fairly simple. They are minimal and by taking s finite étale covering, isomorphic to an abelian surface or a direct product of an elliptic curve and a smooth curve of genus ≥ 2. In this note, we are mainly concerned with the case where X is a smooth projective threefold with non-negative Kodaira dimension $\kappa(X)$. Contrary to the case of algebraic surfaces, they are not necessarily minimal, but similar results also hold in this case. We cannot drop the assumption that $f: X \to X$ is a morphism. There are infinitely many examples which admit a generically finite rational map $f: X \dashrightarrow X$ of degree ≥ 2, eg. a Kummer surface, or a relatively minimal elliptic surface with a global section.

Notations. In the present note, by a smooth projective n-fold X, we mean a non-singular projective manifold of dimension n defined over $\boldsymbol{C}$.
K_X: the canonical bundle of X
$\kappa(X)$: the Kodaira dimension of X
$\chi(\mathcal{O}_X)$: the Euler-Poincaré characteristic of the structure sheaf $\mathcal{O}_X$
$N_1(X) := (\{1\text{-cycles on } X\})/_{\equiv} \otimes_{\boldsymbol{Z}} \boldsymbol{R}$, where $\equiv$ means a numerical equivalence.
$NE(X) :=$ the smallest convex cone in $N_1(X)$ containing all effective 1 - cycles.
$\overline{NE}(X) :=$ Kleiman-Mori cone of X, ie. the closure of $NE(X)$ in $N_1(X)$ for the metric topology.
$\rho(X) := \dim_{\boldsymbol{R}} N_1(X)$, the Picard number of X.

The next propositions, which are direct consequences of Mori theory [1], play a key role in this paper.

Proposition 2. Let $f: Y \to X$ be a finite, étale covering between smooth projective n-folds with $\rho(X) = \rho(Y)$.
Then $f^*: N_1(X) \to N_1(Y)$ (resp. $f_*: N_1(Y) \to N_1(X)$) is isomorphic and $f^* \overline{NE}(X) = \overline{NE}(Y)$ (resp. $f_* \overline{NE}(Y) = \overline{NE}(X)$).
Moreover, if the canonical bundle K_X of X (hence $K_Y \simeq f^* K_X$) is not nef, there is a one to one correspondence between
{extremal rays on $\overline{NE}(X)$} and {extremal rays on $\overline{NE}(Y)$} under the above isomorphisms f^* and f_*.

Proposition 3. Let X, Y be non-singular projective threefolds with non-negative Kodaira dimensions. Assume that X and Y have the same Picard number and there exists a finite étale covering $f: Y \to X$, which is not an isomorphism.
If the canonical bundle K_X of X (hence $K_Y \simeq f^*(K_X)$) is not nef, then Mori's extremal contractions of X (resp. Y), $\mathrm{Cont}_R: X \to X'$ (resp. $\mathrm{Cont}_{\tilde{R}}: Y \to Y'$), associated to each extremal ray R of $\overline{NE}(X)$ (resp. $\tilde{R}$ of $\overline{NE}(Y)$), is a birational divisorial contraction, which is (inverse of) the blow-up along a smooth curve C (resp. $\tilde{C}$) on X' (resp. Y'). Moreover C (resp. $\tilde{C}$) is not $\boldsymbol{P}^1$ and if $f^*(R) = \tilde{R}$, the following commutative diagram holds.

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