On compact conformally flat Einstein-Weyl manifolds

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1. Introduction. Let M be an n-dimensional manifold with a conformal class C. A *conformal connection* on M is an affine connection D preserving the conformal class C, that is, for any Riemannian metric $g \in C$, there exists a 1-form ω_g such that $Dg = \omega_g \otimes g$. We also assume that D is torsion-free. The triple (M, C, D) is called a *Weyl manifold* and D is called a *Weyl structure* on (M, C). A manifold admits an *Einstein-Weyl structure* if there is a Weyl structure for which the symmetric part of the Ricci curvature of the conformal connection is proportional to a metric in C. The Einstein-Weyl equation on the affine connection, which needs an auxilary metric in a given conformal class, is a conformally invariant nonlinear partial differential equation. If (M, g) is an Einstein manifold, then the Levi-Civita connection ∇_g defines an Einstein-Weyl structure of the conformal class $[g]$. Thus the notion of the Einstein-Weyl structure is a generalization of an Einstein metric to conformal structures.

Classically, it is well-known that a conformally flat Einstein manifold must to be a constant curvature manifold. In this paper, as an analogue to this result, we will give classification of closed conformally flat Einstein-Weyl manifolds.

2. Preliminaries. Let (M, C, D) be a Weyl manifold. We assume $n = \dim M \geq 3$. Let Ric^D denote the Ricci curvature of D. In general, Ricci curvature of conformal connection is not symmetric, so we denote by $\mathrm{Sym}(\mathrm{Ric}^D)$ its symmetric part. The scalar curvature R_g^D of D with respect to $g \in C$ is defined by

$$R_g^D = \mathrm{tr}_g \mathrm{Ric}^D. \tag{2.1}$$

A Weyl manifold (M, C, D) is said to be an *Einstein-Weyl manifold* if the symmetric part of the Ricci curvature Ric^D is proportional to the metric g in C. Therefore the *Einstein-Weyl equation* is

$$\mathrm{Sym}(\mathrm{Ric}^D) = \frac{R_g^D}{n} g. \tag{2.2}$$

Note that $R_g^D\, g$ is conformally invariant quantity. In terms of the Ricci curvature and the scalar curvature of the metric $g \in C$, the Einstein-Weyl equation can be written by

$$\mathrm{Ric}_g + \frac{n-2}{4}\left\{\mathcal{L}_{\omega_g^\#} g + \frac{2}{n}(\delta_g \omega_g) g + \omega_g \otimes \omega_g - \frac{|\omega_g|^2}{n} g\right\} = \frac{R_g}{n} g. \tag{2.3}$$

where $\mathcal{L}$ is the Lie derivative, δ_g is the codifferential of g, and the vector field $\omega_g^\#$ is defined as $\omega_g(X) = g(X, \omega_g^\#)$ for all vector fields X.

We prepare some known facts concerning geometry of Weyl manifolds, which we will use in this paper.

Theorem 2.1 (Gauduchon) ([2]). *Let (M, C, D) be a closed Weyl manifold. Then up to homothety, there exists a unique Riemannian metric g in the conformal class C such that the corresponding 1-form ω_g is co-closed: $\delta_g \omega_g = 0$.*

The metric $g \in C$ is called the *Gauduchon metric* if it is up to homothety the unique metric which satisfies $\delta_g \omega_g = 0$.

Corollary 2.2. *Let (M, C, D) be a closed Einstein-Weyl n-manifold, and $g \in C$ the Gauduchon metric. Then $\omega_g^\#$ is a Killing vector field on (M, g), and Einstein-Weyl equation can be written in the following form:*

$$\mathrm{Ric}_g + \frac{n-2}{4}\left(\omega_g \otimes \omega_g - \frac{|\omega_g|^2}{n}\right) = \frac{R_g}{n} g. \tag{2.4}$$

Theorem 2.3 ([4]). *Let (M, C, D) be a connected closed Einstein-Weyl manifold, and $g \in C$ the Gauduchon metric. If the scalar curvature R_g^D of D with respect to g is non-positive but not identically zero, then (M, g) is Einstein.*

Theorem 2.4 ([4]). *Let (M, C, D) be a closed Einstein-Weyl manifold and $g \in C$ the Gauduchon metric. If $R_g^D > 0$, then the fundamental group $\pi_1(M)$ of M is finite.*

Theorem 2.5 ([4]). *Let (M, C, D) be a closed connected non-trivial Einstein-Weyl manifold with $R_g^D = 0$. Then $b_1(M) = 1$.*

Lemma 2.6. *Let (M, C, D) be a connected*