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Delone sets and Riesz basis

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Abstract: In this paper we deal with the density of Delone set and apply it constructing Riesz basis for an Hilbert space.

1. Introduction. A Riesz basis for Hilbert space is easily constructed by exponential maps over a periodic set. This drives us to the question how it is when a periodic set is replaces by Delone set. Construction by exponential functions will certainly work if a Delone set is very close to a periodic set. We are concerned with the problem how a Delone set can be different from the periodic set. In fact, Kadec and Levinson studied such a problem in the case of $L^p[-\pi, \pi]$ (p is a natural number) (see [6] pp. 118-131).

The purpose of the present note is to explore a little further in the cases of $L^2[-\pi, \pi]$ and $H^1[-\pi, \pi]$ (see our main theorem 5.3 and 5.4 below [6]).

2. Delone set and Voronoi cell.

Definition 2.1. An (R, r)-Delone set $\Lambda \subset R^N$ is defined by the next two conditions (see [5] p. 28).

1) Discretness: There exists a positive real number r such that for every $x, y \in \Lambda$, $|x-y| \geq 2r$.
2) Relative density: There is a positive real number R such that every sphere of radius greater than R contains at least one point of Λ in its interior.

Definition 2.2. Let $\Lambda \subset R^N$ be any Delone set. The Voronoi cell at a point $x \in \Lambda$ is the set of points of R^N that lie at least as close to x as to any other point of Λ:

$$V(x) = \{u \in R^N \| x - y| \leq |y-u|, y \in \Lambda\}.$$

The Voronoi cell $V(x)$ is then the smallest convex region about x (see [5] p. 42).

If Λ is a lattice, the Voronoi cells are congruent.

Here we deal with a Delone set Λ including $0: 0 \in \Lambda$.

3. The density of Delone set. We introduce the notion of the density for the (R, r)-Delone set. The density Δ of Delone set Λ centered at x is defined by

$$\Delta_x(\Lambda) = \lim_{s\to\infty} \frac{\#\{\Lambda \cap B_x(s)\}}{m(B_x(s))} \tag{1}$$

(m is the Lebesgue measure).

If (1) is well-defined, we say that Λ has the density $\Delta_x(\Lambda)$ at x.

Here, we should notice that $\Delta_x(\Lambda)$ is actually independent of $x \in R^N$.

Lemma 3.1.

$$\Delta_x(\Lambda) = \Delta_0(\Lambda), \tag{2}$$

for all x.

Proof. Let $\Delta_x(\Lambda)$ be defined for a fixed $x \in R^N$.

Take $s > 0$ such that $s > |x|$. Then

$$B_x(s - |x|) \subset B_0(s) \subset B_x(s + |x|).$$

Here $|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_N^2}$ for $x = (x_1, x_2, \ldots, x_N)$.

Therefore

$$B_x(s-|x|) \cap \Lambda \subset B_0(s) \cap \Lambda \subset B_x(s+|x|) \cap \Lambda.$$

We obtain

$$\frac{\#\{B_x(s-|x|) \cap \Lambda\}}{m(B_0(s))} \leq \frac{\#\{B_0(s) \cap \Lambda\}}{m(B_0(s))} \leq \frac{\#\{B_x(s+|x|) \cap \Lambda\}}{m(B_0(s))}.$$

$$\frac{\#\{B_x(s-|x|) \cap \Lambda\}}{m(B_0(s-|x|))} \left\{\frac{s-|x|}{s}\right\}^N \leq \frac{\#\{B_0(s) \cap \Lambda\}}{m(B_0(s))} \leq \left(\frac{\#\{B_x(s+|x|) \cap \Lambda\}}{m(B_0(s+|x|))}\right) \left\{\frac{s+|x|}{s}\right\}^N$$

($\#$ is the number of elements).

We have (2) when $s \to \infty$. □

Corollary 3.2.

$$\Delta_x(\Lambda) = \Delta_y(\Lambda),$$

for $x \neq y$.

We now define the density of Λ by

$$\Delta(\Lambda) = \lim_{s\to\infty} \frac{\#\{\Lambda \cap B_0(s)\}}{m(B_0(s))} \tag{3}$$

and call $\Delta(\Lambda)$ the density of Λ.