On a Relation Among Toric Minimal Models

By Daisuke MATSUSHITA

Resarch Institute of Mathematical Science, Kyoto University
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Abstract: Every normal surface singularity has a unique minimal resolution. On the contrary, a minimal terminalization of higher dimensional singularity is not unique. In this note, we prove that there exists a correspondence between minimal terminalizations of a toric canonical singularity and radicals of initial ideals of term order represented by weight vector.

1. Introduction. Every normal surface singularity has a uniquely determined good resolution called minimal resolution, which plays an inportant role in the studying of surface singularities. For singularities of higher dimension, Minimal Model Conjecture tells us that there should exist a *minimal terminalization.*

Definition. A *minimal terminalization* of a germ of singularities X is a projective birational morphism $\pi : Y \rightarrow X$ which satisfies the following two conditions:

(1) Y has only $\boldsymbol{Q}$-factorial terminal singularities.

(2) $K_Y \sim \pi^* K_X + \sum a_i E_i, \quad a_i \leq 0.$

We say that π is a *minimal resolution* or a *minimal* $\boldsymbol{Q}$*-factorization* if Y is smooth or has only $\boldsymbol{Q}$-factorial canonical singularities, respectively.

It is known that three dimensional singularities and toric singularities have a minimal terminalization. Minimal terminalizations have many nice properties like as minimal resolutions of surface singularities. In dimension three or higher, however, a minimal terminalization is not unique. In this note, we prove that there exists a correspondence between minimal terminalizations of a toric canonical singularity and radicals of initial ideals of term order represented by weight vector.

Definition. Let $R = \boldsymbol{C}[x_1, \cdots, x_n]$ be a polynomial ring in n variables. Fix $\omega = (\omega_1, \cdots, \omega_n) \in \boldsymbol{R}^n$. For any polynomial $f = \sum c_i x^{\alpha_i}$, we define the *initial form* $\mathrm{in}_\omega(f)$ to be the sum of all terms such that the inner product $\omega \cdot \alpha_i$ is maximal. The *initial ideal* attached to a given ideal I is defined to be the ideal generated by all the initial forms:

$$\mathrm{in}_\omega(I) := \langle \mathrm{in}_\omega(f) : f \in I \rangle.$$

We notice that this ideal is not necessarily to be a monomial ideal.

Our main theorem is the following.

Theorem 1. *Let X be a d-dimensional toric canonical singularity. Then there exists an homogeneous binomial ideal I of $\boldsymbol{C}[x_1, \cdots, x_n]$ which satisfies the following four conditions:*

(1) *The ideal I defines the toric variety defined by the dual fan of the defining of X.*

(2) *There exists a one-to-one correspondence between the minimal $\boldsymbol{Q}$-factorizations and the radicals of initial ideals of weight ω in I such that $\mathrm{Rad}(\mathrm{in}_\omega(I))$ is a monomial ideal.*

(3) $\mathrm{Rad}(\mathrm{in}_\omega(I))$ *corresponds to the minimal terminalization of and only if $\mathrm{Rad}(\mathrm{in}_\omega(I))$ does not contain $(1 \leq i \leq n)$.*

(4) *If X is a Gorenstein canonical singularity, $\mathrm{Rad}(\mathrm{in}_\omega(I))$ corresponds to the minimal resolution if and only if $\mathrm{Rad}(\mathrm{in}_\omega(I)) = \mathrm{in}_\omega(I)$.*

2. Proof of theorem. Let $X = \mathrm{Spec}\,\boldsymbol{C}[\sigma^\vee \cap M]$. Assume that the cone σ is generated by $a_1, \cdots, a_m$. Because X has only canonical singularity, by [5, 1.11], there exists a linear function h such that $h(a_i) = r\ (1 \leq i \leq m)$ and $h(b) \geq r$ for $b \in \sigma \cap N$, where r is a positive integer. Let Δ be a $d-1$-dimensional integral polygon such that

$$\Delta := \{x \in \sigma \mid h(x) = r\}.$$

We define a *regular triangulation* of integral polytope.

Definition. Let Δ be a $d-1$-dimensional

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