# On Jeśmanowicz' Conjecture Concerning Pythagorean Numbers*),**)

By Maohua LE

Department of Mathematics, Zhanjiang Teachers College, Guangdong P. R. China

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**Abstract**: Let $r$, $s$ be positive integers satisfying $r > s$, $2 \mid r$ and $\gcd(r, s) = 1$. In this paper, using Baker's method, we prove that if $2 \parallel r$, $r \geq 81s$ and $s \equiv 3 \pmod 4$, then the equation $(r^2 - s^2)^x + (2rs)^y = (r^2 + s^2)^z$ has the only solution $(x, y, z) = (2,2,2)$.

**Key words and phrases**: Exponential diophantine equation; Jeśmanowicz' conjecture; Baker's method.

**1. Introduction.** Let $\boldsymbol{Z}$, $\boldsymbol{N}$, $\boldsymbol{Q}$ be the sets of integers, positive integers and rational numbers, respectively. Let $(a, b, c)$ be a primitive Pythagorean triple such that

$$(1) \qquad a^2 + b^2 = c^2,\ a, b, c \in \boldsymbol{N}, \quad \gcd(a, b, c) = 1,\ 2 \mid b.$$

Then we have, as is well known,

$$(2) \qquad a = r^2 - s^2,\ b = 2rs,\ c = r^2 + s^3,$$

where $r$, $s$ are positive integers satisfying $r > s$, $\gcd(r, s) = 1$ and $2 \mid rs$. In [2], Jeśmanowicz conjectured that the only solution of the equation

$$(3) \qquad a^x + b^y = c^z, \quad x, y, z \in \boldsymbol{N}$$

is $(x, y, z) = (2,2,2)$. This conjecture was proved for some special cases (see the references of [4]). But, in general, the problem is not solved as yet. Recently, Takakuwa and Asaeda [6] proved that if $2 \parallel r$, $s = 3$ and $r$ satisfies some other conditions, then the only solution of (3) is $(x, y, z) = (2,2,2)$. Guo and Le [1] showed that the conditions on $r$ can be reduced to $r \geq 6000$, improving the result of [6]. In this paper we prove a general result as follows:

**Theorem.** If $2 \parallel r$, $s \equiv 3 \pmod 4$ and $r \geq 81s$, then the only solution of (3) is $(x, y, z) = (2,2,2)$.

By this theorem, the above condition $r \geq 6000$ in the result of [1] can be replaced by $r \geq 243$.

**2. Preliminaries. Lemma 1.** ([5, page 2]). The equation

$$X^4 + Y^2 = Z^4,\ X, Y, Z \in \boldsymbol{N}$$

has no solution $(X, Y, Z)$.

**Lemma 2.** ([1, Lemma 2]). Let $(x, y, z)$ be a solution of (3) with $(x, y, z) \neq (2,2,2)$. If $2 \parallel r$ and $s \equiv 3 \pmod 4$, then we have $2 \mid x$, $y = 1$ and $2 \nmid z$.

Let $\alpha$ be a nonzero algebraic number with the defining polynomial $a_0 z^n + a_1 z^{n-1} + \cdots + a_n = a_0(z - \sigma_1\alpha)\cdots(z - \sigma_n\alpha)$, where $a_0 \in \boldsymbol{N}$, $\sigma_1\alpha, \cdots, \sigma_n\alpha$ are all the conjugates of $\alpha$. Then

$$h(\alpha) = \frac{1}{n}\Big(\log a_0 + \sum_{i=1}^{n} \log\max(1, |\delta_i\alpha|)\Big)$$

is called Weil's height of $\alpha$.

**Lemma 3.** Let $\alpha_1$, $\alpha_2$ be positive real algebraic numbers which are multiplicatively independent. Further let $D = [\boldsymbol{Q}(\alpha_1, \alpha_2) : \boldsymbol{Q}]$ and $\log A_j = \max(h(\alpha_j), |\log \alpha_j|/D, 1/D)$ for $j = 1,2$. Let $\Lambda = b_1 \log \alpha_1 - b_2 \log \alpha_2$, $b_1, b_2 \in \boldsymbol{N}$. Then we have

$$\log|\Lambda| \geq -32.31\, D^4 (\log A_1)(\log A_2) \cdot \Big(\max\Big(\frac{10}{D}, 0.18 + \log B\Big)\Big)^2,$$

where $B = b_1/D \log A_2 + b_2/D \log A_1$.

*Proof.* Letting $h_2 = 10$ in the Table 2 in [3], we obtain this lemma immediately in the same way as Corollary 2 of [3].

**3. Proof of theorem.** We now assume that $r$ and $s$ satisfy $2 \parallel r$, $s \equiv 3 \pmod 4$ and $r \geq 81s$. Let $(x, y, z)$ be a solution of (3) with $(x, y, z) \neq (2,2,2)$. Then, by Lemma 2, we have

$$(4) \qquad a^x + b = c^z,\ 2 \mid x,\ 2 \nmid z.$$

Further, by the proof of [1, Theorem], we have $z < x$.

Since $c = a + 2s^2$, we get from (2) that

$$(5) \qquad \log c = \log a + \rho_1,$$

where $\rho_1$ satisfies

$$(6) \qquad 0 < \rho_1 = \frac{2s^2}{r^2} \sum_{k=0}^{\infty} \frac{1}{2k+1}\Big(\frac{s^2}{r^2}\Big)^{2k}$$

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