Commutant Algebra of Superderivations on a Grassmann Algebra

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Introduction. In his classical book [4], Weyl gave the constructions of the representations of the general linear groups by using Young's symmetrizers. Undoubtedly his theory is very important in the representation theory and many successors have worked in the generalization of this theory. We also try to get a similar construction for natural representations of Cartan-type Lie algabras and Cartan-type Lie superalgabras. As the first step in this direction, it seems necessary to calculate the commutant algebra of this representation. For the case of Cartan-type Lie algebra of vector fields, the first author successfully found the commutant algebra for the case $m \leq n$ (see [2]), where m is the power of tensor product and n is the rank of the Lie algebra. In this article, we want to look for the commutant algebra of the natural representation of Catan-type Lie *superalgebra* $W(n)$ consisting of all the superderivations on the Grassmann algebra of n-variables (see below for the definition). For the case $m \leq n$ (here also m is the power of tensor product), using the same method as in [2], we obtain the result (see Section 2). For the case $m > n$, it seems more complicated, but for $n = 1$ and arbitrary m, we get the similar result as in the case $m \leq n$; furthermore, in this case we also get the bicommutant algebra (see Section 3). For the general case, we conjecture that the result is the same as in the case $m \leq n$. As an evidence, in Section 4, we give an example for the case $n = 2$, $m = 3$.

1. Lie superalgebra $W(n)$ and its natural representation. Let $\Lambda(n)$ be a Grassmann algebra over $\boldsymbol{C}$ in n variables $\xi_1, \xi_2, \cdots, \xi_n$ and Λ_k be the space of k-homogeneous elements of $\Lambda(n)$. Put $\Lambda(n)_{\bar{0}} := \sum_{k:\text{even}} \Lambda_k$ and $\Lambda(n)_{\bar{1}} := \sum_{k:\text{odd}} \Lambda_k$, then $\Lambda(n)$ has a natural $\boldsymbol{Z}_2$-grading and so we consider $\Lambda(n)$ as a superalgebra. Let $W(n)$ be the set of all the superderivations over $\Lambda(n)$, then it becomes naturally a Lie superalgebra. According to the results in [1], every superderivation $D \in W(n)$ can be written in the form $D = \sum_{i=1}^{n} P_i \frac{\partial}{\partial \xi_i}$ with $P_i \in \Lambda(n)$ $(1 \leq i \leq n)$, where $\frac{\partial}{\partial \xi_i}$ is a superderivation of degree 1 defined by $\frac{\partial}{\partial \xi_i} \xi_j = \delta_{ij}$. By definition, the Lie superalgebra $W(n)$ acts on Grassman algebra $\Lambda(n)$ as follows: for any homogeneous $D \in W(n)$ and $\forall\, \xi_{i_1} \wedge \cdots \wedge \xi_{i_r}$,

$$D(\xi_{i_1} \wedge \cdots \wedge \xi_{i_r}) = \sum_{s=1}^{r} (-1)^{(s-1)\deg D} \xi_{i_1} \wedge \cdots \wedge D(\xi_{i_s}) \wedge \cdots \wedge \xi_{i_r}.$$

We call it a *natural representation* of $W(n)$, and denote it by ϕ.

Let us consider m-fold tensor product $\otimes^m \Lambda(n)$. Then we have a natural isomorphism as $W(n)$-modules

$$\otimes^m \Lambda(n) \simeq \Lambda[\xi_{ij} \mid 1 \leq i \leq n,\ 1 \leq j \leq m] =: \Lambda(n, m),$$

where $\Lambda[\xi_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq m]$ is a Grassmann algebra generated by ξ_{ij} $(1 \leq i \leq n, 1 \leq j \leq m)$. In the following, we identify $\otimes^m \Lambda(n)$ with $\Lambda(n, m)$. By means of a tensor product, $W(n)$ is imbedded into $\mathrm{End}(\otimes^m \Lambda(n)) \cong \mathrm{End}\Lambda(n, m)$. More precisely, an element

$$D = \sum_{i=1}^{n} P_i(\xi_1, \cdots, \xi_n) \frac{\partial}{\partial \xi_i} \in W(n)$$

corresponds to an element

$$\phi^{\otimes m}(D) = \sum_{i=1}^{n} \sum_{\alpha=1}^{m} P_i(\xi_{1\alpha}, \cdots, \xi_{n\alpha}) \frac{\partial}{\partial \xi_{i\alpha}} \in \mathrm{Der}\Lambda(n, m)$$

via m-fold tensor product $\phi^{\otimes m}$ of ϕ.

Let $\mathscr{C}_m$ denote the commutant algebra of $\phi^{\otimes m}(W(n))$ in $\mathrm{End}(\Lambda(n, m))$:

$$\mathscr{C}_m = \{E \in \mathrm{End}(\otimes^m \Lambda(n)) \mid [E, D] = 0,$$

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