On $L(1,\chi)$ and Class Number Formula for the Real Quadratic Fields

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1. Introduction. Let k be a positive integer greater than 1, and let $\chi(n)$ be a real primitive character modulo k. The series

$$L(1, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n}$$

can be divided into groups of k consecutive terms. Let v be any nonnegative integer, j an integer, $0 \leq j \leq k-1$, and let

$$T(v, j, \chi) = \sum_{n=j+1}^{j+k} \frac{\chi(vk+n)}{vk+n} = \sum_{n=j+1}^{j+k} \frac{\chi(n)}{vk+n}.$$

Then $L(1, \chi) = \sum_{n=1}^{j} \frac{\chi(n)}{n} + \sum_{v=0}^{\infty} T(v, j, \chi)$.

We remind the reader that a real primitive character $(\mathrm{mod}\, k)$ exists only when either k or $-k$ is a fundamental discriminant, and that the character is then given by

$$\chi(n) = \left(\frac{d}{n}\right),$$

where d is k or $-k$, and the symbol is that of Kronecker (see, for example, Ayoub [2] for the definition of a Kronecker character).

In [5], Davenport proved the following theorem:

Theorem (H. Davenport). *If* $\chi(-1) = 1$, *then* $T(v, 0, \chi) > 0$ *for all* v *and* k. *If* $\chi(-1) = -1$, *then* $T(0,0, \chi) > 0$ *for all* k, *and* $T(v, 0, \chi) > 0$ *if* $v > v(k)$: *but for any integer* $r \geq 1$ *there exist values of* k *for which*

$$T(1,0, \chi) < 0,\ T(2,0, \chi) < 0, \ldots,\ T(r, 0, \chi) < 0.$$

In [9], Leu and Li derived the following result about $T\left(v, \left[\frac{k}{2}\right], \chi\right)$.

Theorem A. *If* $\chi(-1) = 1$, *then* $T\left(v, \left[\frac{k}{2}\right], \chi\right) < 0$ *for all* v *and* k, *where* $[x]$ *denotes the greatest integer* $\leq x$.

Combining the results of Davenport [5] and Theorem A of Leu and Li, we have the following interesting inequalities.

Theorem B. *If* $\chi(-1) = 1$, *then*

$$\sum_{n=1}^{k} \frac{\chi(n)}{n} < L(1, \chi) < \sum_{n=1}^{[\frac{k}{2}]} \frac{\chi(n)}{n}. \tag{1.1}$$

In section 2, we will prove that $T(v, j, \chi) \neq 0$ for prime integer $k > 2$, nonnegative integer v and $j = 0,1,2, \ldots, k-1$. In section 3, we will derive the inequalities for $L(1, \chi)$ on even real primitive character modulo k:

$$\sum_{n=1}^{[\frac{k}{2}]} \frac{\chi(n)}{n} - \frac{\ln k}{\sqrt{k}} < L(1, \chi) < \sum_{n=1}^{[\frac{k}{2}]} \frac{\chi(n)}{n}. \tag{1.2}$$

On the other hand, using Siegel-Tatuzawa's lower bound for $L(1, \chi)$ (See [15] and [13]) and Louboutin's upper bound for $L(1, \chi)$ (See [10], [3], [14] and [12]), one also has inequalities for $L(1, \chi)$ on even real primitive character modulo k (with one possible exception coming from applying Siegel-Tatuzawa's theorem [15]):

$$\frac{0.655}{4} \frac{1}{k^{1/4}} < L(1, \chi) \leq \frac{1}{2} \ln k + \frac{2 + \gamma - \ln(4\pi)}{2}, \tag{1.3}$$

where $k \geq e^{11.2}$ and γ denotes Euler's constant. From the facts $\lim_{k\to\infty} \frac{\ln k}{\sqrt{k}} = 0$ and $\lim_{k\to\infty}\left(\frac{1}{2}\ln\, k - \frac{0.655}{4} k^{-\frac{1}{4}}\right) = \infty$, it is clear that the inequalities (1.2) provides much better estimate for $L(1, \chi)$ than the inequalities (1.3) does. In these days, computing facilities are highly developed, the inequalities (1.2) may be used to investigate varied problems related to $L(1, \chi)$. In section 4, we will derive a class number formula for the real quadratic fields:

- If positive integer k is not of the form $m^2 + 4$ ($m \in \mathbf{N}$), then

$$h = \left[\frac{\sqrt{k}}{2\ln \varepsilon} \sum_{n=1}^{[\frac{k}{2}]} \frac{\chi(n)}{n}\right],$$

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