Power Series with the Riemann Zeta-function in the Coefficients

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1. Introduction. Let $\zeta(s)$ be the Riemann zeta-function, and $\zeta(s, \alpha)$ with a parameter $\alpha > 0$ the Hurwitz zeta-function defined by

$$\zeta(s, \alpha) = \sum_{n=0}^{\infty} (n+\alpha)^{-s} \quad (\mathrm{Re}\, s > 1),$$

and its meromorphic continuation over the whole s-plane. Let $\Gamma(s)$ be the gamma-function, and $(s)_n = \Gamma(s+n)/\Gamma(s)$ for any integer n Pochhammer's symbol.

The main aim of this note is to investigate two types of power series whose coefficients involve the Riemann zeta-function (see Sections 2 and 3) based on Mellin-Barnes' type integral formulae. Further, as for generalizations of these power series, we shall introduce hypergeometric type generating functions of $\zeta(s)$ and derive their basic properties in the final section. Proofs of the results in the following sections are only sketched. Detailed version of the proofs will appear in a forthcoming paper.

2. Binomial type series. A simple relation

$$\sum_{n=2}^{\infty} \{\zeta(n) - 1\} = 1,$$

which was firstly mentioned by Goldbach in 1729 (see [10, Section 1]), follows immediately from the inversion of the order of the double sum $\sum_{n=2}^{\infty} \sum_{m=2}^{\infty} m^{-n}$. This is in fact derived as a special case of Ramanujan's formula

$$\zeta(\nu, 1+x) = \sum_{n=0}^{\infty} \frac{(\nu)_n}{n!} \zeta(\nu+n)(-x)^n \tag{2.1}$$

for $|x| < 1$ and any complex $\nu \notin \{-1,0,1,2,\ldots\}$, which gives a base of his various evaluations of sums involving $\zeta(s)$ (see [7, Sections 5 and 6]). Noting the relations $\zeta(s, 1) = \zeta(s)$ and $(\partial/\partial\alpha)^n \zeta(s, \alpha) = (-1)^n (s)_n \zeta(s+n, \alpha)$, we see that (2.1) is actually the Taylor series expansion of $\zeta(\nu, 1+x)$ as a function of x near $x = 0$. Srivastava [9][10] proved various summation formulae related to (2.1), while Klusch [6] considered a generalization of (2.1) to the Lerch zeta-function. This direction has recently been pursued by Yoshimoto, Kanemitsu, and the author [15]. Rane [8] applied (2.1) to study the mean square of Dirichlet L-functions.

For our later purpose we shall prove (2.1) as an application of Mellin-Barnes' type integrals. Suppose first that $\mathrm{Re}\, \nu > 1$, and set

$$F_\nu(x) = \frac{1}{2\pi i} \int_{(b)} \frac{\Gamma(\nu+s)\Gamma(-s)}{\Gamma(\nu)} \zeta(\nu+s) x^s ds \tag{2.2}$$

for $x > 0$, where b is fixed with $1 - \mathrm{Re}\, \nu < b < 0$, and (b) denotes the vertical straight line from $b - i\infty$ to $b + i\infty$. We can shift the path of integration in (2.2) to the right, provided $0 < x < 1$. Collecting the residues at the poles $s = 0,1,2,\ldots$ of the integrand, we see that $F_\nu(x)$ is equal to the right-hand infinite series in (2.1). On the other hand, since $\zeta(\nu+s) = \sum_{n=1}^{\infty} n^{-\nu-s}$ converges absolutely on the path $\mathrm{Re}\, s = b$, the term-by-term integration is permissible, and this gives

$$F_\nu(x) = \sum_{n=1}^{\infty} (n+x)^{-\nu} = \sum_{n=0}^{\infty} (n+1+x)^{-\nu},$$

where each term in the resulting expression could be evaluated by taking $-z = x/n$ in

$$\Gamma(a)(1-z)^{-a} = \frac{1}{2\pi i} \int_{(\sigma)} \Gamma(a+s)\Gamma(-s)(-z)^s ds$$

for $|\arg(-z)| < \pi$ and $-\mathrm{Re}\, a < \sigma < 0$ (cf. [14], p. 289, 14.51, Corollary]). We therefore obtain (2.1) by analytic continuation.

3. Exponential type series. Chowla and Hawkins [2] found that the sum

$$G_0(x) = \sum_{n=2}^{\infty} \zeta(n) \frac{(-x)^n}{n!}$$

has the asymptotic formula

$$G_0(x) = x \log x + (2\gamma - 1)x + \frac{1}{2} + O(e^{-A\sqrt{x}}) \tag{3.1}$$

as $x \to +\infty$, where γ is Euler's constant and A is a certain positive constant. They conjectured that the error term in (3.1) cannot be essentially sharpened. Let a be an arbitrary fixed real number. Buschman and Srivastava [1] introduced a

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