

# On the Degrees of Irrationality of Hyperelliptic Surfaces

By Hisao YOSHIHARA

Faculty of Science, Niigata University

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1. Let  $L$  be a field, which is a finitely generated extension of a ground field  $k$ , and assume that  $\text{tr.deg.}_k L = n$ . We denote by  $d_r(L)$  the degree of irrationality of  $L$  over  $k$ , which is defined to be the number (cf. [2], [5]):

$$\min \left\{ m \left| \begin{array}{l} m = [L : k(x_1, \dots, x_n)], \text{ where } x_1, \dots, \\ x_n \text{ are algebraically independent ele-} \\ \text{ments of } L. \end{array} \right. \right\}$$

We call the field  $k(x_1, \dots, x_n)$ , which defines the value  $d_r(L)$ , a maximal rational subfield of  $L$  and write *m.r.subf.* for short. For an algebraic variety  $V$  defined over  $k$ , we define the degree of irrationality of  $V$  to be  $d_r(k(V))$ , where  $k(V)$  is the rational function field of  $V$ . Clearly it is a birational invariant of algebraic varieties. In other words it is the minimal degree of a dominant rational map from  $V$  to the projective  $n$ -space. Hence, when  $n = 1$ , it coincides with the gonality of a curve. In case  $k$  is not algebraically closed, for example  $k = \mathbb{Q}$ , we feel a great interest in the value  $d_r$ . Because,  $d_r$  seems to have some relations with the least number  $[k' : k]$  such that the variety  $V$  has many rational points over  $k'$  (see, e.g. [1]). But it is very difficult to find this value. We assume that  $k = \mathbb{C}$  hereafter. In this note we announce the results for  $d_r(S)$  of hyperelliptic surfaces  $S$ . Details will appear elsewhere.

2. Let  $S$  denote a hyperelliptic surface. Of course we have that  $d_r(S) \geq 2$ . First we give examples.

**Example 1.** Let  $A_i$  ( $i = 1, 2$ ) be the abelian surface defined by the following period matrix:

$$\Omega_1 = \begin{pmatrix} 1 & 0 & \alpha & 0 \\ 0 & 1 & 0 & \beta \end{pmatrix} \text{ or } \Omega_2 = \begin{pmatrix} 1 & 0 & \alpha & 0 \\ 0 & 1 & 1/2 & \beta \end{pmatrix},$$

where  $\Im \alpha \neq 0$  and  $\Im \beta \neq 0$ . Let  $g$  be the automorphism of  $A_i$  defined by

$$g(z_1, z_2) = (z_1 + 1/2, -z_2).$$

Then  $g^2 = \text{id}$  on  $A$  and  $S_i = A_i/g$  is a hyperelliptic surface. Moreover letting  $h(z_1, z_2) = (-z_1, z_2)$ , we see that  $h$  defines an automorphism of  $S_i$  and  $S_i/h$  becomes a rational

surface. Note that  $A_i/h$  and  $A_i/gh$  are (birationally equivalent to) a ruled surface with irregularity 1 and a K3 surface respectively (cf. [6]).

Let  $K_S$  and  $\sim$  denote the canonical divisor of  $S$  and the linear equivalence of divisors respectively. Then we have the following

**Lemma 2.** Suppose that there is an automorphism  $\varphi$  of  $S$  with an order  $d (\neq 1)$  such that  $S/\varphi$  is rational. Then  $d = 2, 3, 4$  or  $6$ , and moreover the following facts hold true:

- (1) If  $d = 2$  or  $4$ , then  $2K_S \sim 0$ .
- (2) If  $d = 3$ , then  $3K_S \sim 0$ .
- (3) If  $d = 6$ , then  $2K_S \sim 0$  or  $3K_S \sim 0$ .

Using this lemma, we obtain the following

**Theorem 3.**  $d_r(S) = 2$  if and only if  $2K_S \sim 0$ , i.e.,  $S$  is isomorphic to one of the surfaces in Example 1.

Before considering other surfaces, we present some more examples.

**Example 4.** Let  $A_i$  ( $i = 1, 2$ ) be the abelian surface defined by the following period matrix:

$$\Omega_1 = \begin{pmatrix} 1 & 0 & \omega & 0 \\ 0 & 1 & 0 & \omega \end{pmatrix} \text{ or } \Omega_2 = \begin{pmatrix} 1 & 0 & (\omega - 1)/3 & 0 \\ 0 & 1 & (\omega - 1)/3 & \omega \end{pmatrix},$$

where  $\omega = \exp(2\pi\sqrt{-1}/3)$ . Let  $g_i$  be the automorphism of  $A_i$  defined by

$$g_1 z = (z_1 + (\omega + 2)/3, \omega z_2) \text{ and } g_2 z = (z_1 + 1/3, \omega z_2),$$

where  $z = (z_1, z_2)$ . Then  $g_i^3 = \text{id}$  on  $A$  and  $S_i = A_i/g_i$  is a hyperelliptic surface. Moreover letting  $h_1 z = \omega z$  and  $h_2 z = (\omega z_1, \omega z_2 + 2/3)$ ,

we see that  $h_i$  defines an automorphism of  $S_i$  and  $S_i/h_i$  becomes a rational surface. Note that  $A/h$ ,  $A/gh$  and  $A/gh^2$  are (birationally equivalent to) a rational surface, a K3 surface and a ruled surface with irregularity 1 respectively, where we put  $A = A_i$ ,  $g = g_i$  and  $h = h_i$ .

These examples are unique in the following sense.

**Theorem 5.** For hyperelliptic surfaces  $S$  the following conditions (i) and (ii) are equivalent: