On Subsets of C^{n+1} in General Position

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1. Introduction. Let X be a subset of C^{n+1} such that $\# X \geq n+1$. It is said that X is in general position if any $n+1$ elements of X are linearly independent. From now on throughout the paper we suppose that X is in general position. Put

$$X(0) = \{\boldsymbol{a} = (a_1, \ldots, a_n, a_{n+1}) \in X : a_{n+1} = 0\} \text{ and } \nu = \# X(0).$$

It is easily seen that $0 \leq \nu \leq n$. We introduce the following notion to refine the fundamental inequality of H. Cartan for holomorphic curves ([1]).

Definition 1. We say that

(i) X is maximal (in the sense of general position) if and only if for any Y in general position such that $X \subset Y \subset C^{n+1}$, $X = Y$.

(ii) X is ν-maximal if X is maximal and $\# X(0) = \nu$.

The purpose of this paper is to give an example of ν-maximal subset of C^{n+1} for any $\nu (1 \leq \nu \leq n)$. Applications to the value distribution theory of holomorphic curves ([2], [3]) will appear elsewhere.

2. Lemma. We use the following notation.

(a) The difference product of $x_1, \ldots, x_n$:

$$\varDelta_n = \varDelta(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n} (x_i - x_j) = \begin{vmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & x_1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ x_n^{n-1} & x_n^{n-2} & \cdots & x_n & 1 \end{vmatrix}.$$

(b) The elementary symmetric polynomials in $x_1, \ldots, x_n$:

$$\sigma_{n0} = \sigma_0(x_1, \ldots, x_n) = 1,$$
$$\sigma_{n1} = \sigma_1(x_1, \ldots, x_n) = x_1 + \cdots + x_n,$$
$$\sigma_{n2} = \sigma_2(x_1, \ldots, x_n) = x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n,$$
$$\cdots\cdots$$
$$\sigma_{nn-1} = \sigma_{n-1}(x_1, \ldots, x_n) = x_1 \cdots x_{n-1} + x_1 \cdots x_{n-2}x_n + \cdots + x_2 \cdots x_n,$$
$$\sigma_{nn} = \sigma_n(x_1, \ldots, x_n) = x_1 \cdots x_n,$$
$$\sigma_{nn+1} = \sigma_{n+1}(x_1, \ldots, x_n) = 0.$$

(c) We put

$$f_{nj}(t) = \sigma_j(1, t, \ldots, t^{n-1}) \ (j = 0, 1, \ldots, n+1).$$

These f_{nj} are polynomials, which are not identically equal to zero except f_{nn+1}.

Lemma 1. For $j = 1, \ldots, n$,

$$\sigma_j(x_1, \ldots, x_n) = \sigma_j(x_1, \ldots, x_{n-1}) + \sigma_{j-1}(x_1, \ldots, x_{n-1})x_n.$$

This is easily seen from the definition of the elementary symmetric polynomials.

Lemma 2.

$$(1) \quad \begin{vmatrix} \alpha_1 & \alpha_2 & \cdots\cdots & \alpha_n & \alpha_{n+1} \\ x_1^n & x_1^{n-1} & \cdots\cdots & x_1 & 1 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ x_n^n & x_n^{n-1} & \cdots\cdots & x_n & 1 \end{vmatrix}$$

$$= \varDelta_n \sum_{j=1}^{n+1} (-1)^{j-1} \sigma_{nj-1} \alpha_j$$

$$= \varDelta_n \sum_{j=1}^{n} (-1)^{j-1} \sigma_{n-1j-1} \cdot (\alpha_j - x_n \alpha_{j+1})$$

The first equality is well-known and we can prove the second one by Lemma 1.

Let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n+1}$ be the standard basis of C^{n+1}.

Lemma 3. For any vector $(\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$ of C^{n+1} which is not equal to 0, $\alpha \boldsymbol{e}_1$, or $\beta \boldsymbol{e}_{n+1}$, there exist complex numbers $a_1, \ldots, a_n$ different from each other for which the vectors

$$(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}), \ (a_1^n, \ldots, a_1, 1), \ldots, (a_n^n, \ldots, a_n, 1)$$

are linearly dependent, where α and β are any complex numbers.

Proof. We have only to find $x_j = a_j (j = 1, \ldots, n)$ different from each other for which the determinant (1) reduces to zero.

(a) The case when at least two of $\alpha_1, \ldots, \alpha_{n+1}$ are different from zero.

Put $x_k = t^{k-1}x_1 \ (k = 2, \ldots, n)$ and substitute them into (1). Then

$$\sigma_j(x_1, \ldots, x_n) = x_1^j f_{nj}(t) \ (j = 1, \ldots, n)$$

and the right-hand side of (1) divided by $\varDelta_n$ is equal to

$$(2) \quad \alpha_1 - \alpha_2 f_{n1}(t)x_1 + \cdots + (-1)^{n-1}\alpha_n f_{nn-1}(t)x_1^{n-1} + (-1)^n \alpha_{n+1} f_{nn}(t)x_1^n.$$

Let $t = t_0 (\neq 0)$ be any number for which