On the Non-Analytic Examples of Christ and Geller

By Joe KAMIMOTO

Department of Mathematical Sciences, University of Tokyo
(Communicated by Kiyosi ITÔ, M. J. A., March 12, 1996)

1. Introduction. In [3], M. Christ and D. Geller gave the following remarkable counterexample to analytic hypoellipticity of $\bar{\partial}_b$ for real analytic CR manifolds of finite type:

Theorem 1.1. *On the three-dimensional CR manifold $M_m := \{\mathrm{Im} z_2 = [\mathrm{Re} z_1]^{2m}\}$ $(m = 2,3,\ldots)$, $\bar{\partial}_b$ fails to be relatively analytic hypoelliptic.*

Here, *relative analytic hypoellipticity* of $\bar{\partial}_b$ is a notion different from the usual one. Its definition is given in §2.

Moreover, relative analytic hypoellipticity of $\bar{\partial}_b$ is closely connected with real analyticity of the Szegö kernel off the diagonal. By considering the Szegö kernel as a singular solution of the equation $\bar{\partial}_b u = 0$, Christ and Geller obtained Theorem 1.1 as a corollary of the following theorem.

Theorem 1.2. *The Szegö kernel of $M_m (m = 2,3,\ldots)$ fails to be real analytic off the diagonal.*

The proof of Theorem 1.2 by Christ and Geller [3] is based on certain formulas of Nagel [5]. Though their proof is logically clear, it seems to be difficult to understand the singularity of the Szegö kernel of M_m directly. On the other hand, M. Christ [1] constructed singular solutions for $\bar{\partial}_b$ directly, and proved Theorem 1.1. In this note, we give the Szegö kernel of M_m an integral representation in terms of the singular solutions of Christ. Since the singular solutions of Christ are substantially simpler, our representation makes it easy to understand the singularity of the Szegö kernel of M_m. We also give a similar representation to the Bergman kernel of the domain $\{\mathrm{Im} z_2 > [\mathrm{Re} z_1]^{2m}\} \subset \boldsymbol{C}^2 (m = 2,3,\ldots)$ on the boundary.

Finally we remark that our subject in this note has also been treated in the paper of M. Christ ([2], §7), and our result can be considered as an improvement of Proposition 7.2 in [2].

2. Statement of our results. Consider the hypersurface $M := \{\mathrm{Im} z_2 = P(z_1)\} \subset \boldsymbol{C}^2$, where $P : \boldsymbol{C} \to \boldsymbol{R}$ is a subharmonic, nonharmonic polynomial. Such a surface is pseudoconvex and of finite type. A nonvanishing, antiholomorphic, tangent vector field is $\partial/\partial \bar{z}_1 - 2i(\partial P/\partial \bar{z}_1)\partial/\partial \bar{z}_2$. As coordinates for the surface we use $\boldsymbol{C} \times \boldsymbol{R} \ni (z = x + iy, t) \mapsto (z, t + iP(z))$; the vector field pulls back to $\bar{\partial}_b = \partial/\partial \bar{z} - i(\partial P/\partial \bar{z})\partial/\partial t$. Let $\bar{\partial}_b^*$ denote the formal adjoint of $\bar{\partial}_b$ with respect to the Lebesgue measure on $\boldsymbol{C} \times \boldsymbol{R}$. Recall the natural notion of analytic hypoellipticity for $\bar{\partial}_b$:

Definition 2.1. *$\bar{\partial}_b$ is relatively analytic hypoelliptic on M, if whenever $\bar{\partial}_b u$ is real analytic in an open set V and $u = \bar{\partial}^*{}_b v$ for some $v \in L^2$ in V, u is real analytic in V.*

In usual sense, $\bar{\partial}_b$ is not even C^∞ hypoelliptic, but it is well-known that if $\bar{\partial}_b \bar{\partial}_b^* u \in C^\infty$, then $\bar{\partial}_b^* u \in C^\infty$ ([4]).

When $M = M_m = \{\mathrm{Im} z_2 = [\mathrm{Re} z_1]^{2m}\}$ $(m = 2,3,\ldots)$, M. Christ in [1], [2] constructed the singular solutions of the equation $\bar{\partial}_b u = 0 (u = \bar{\partial}_b^* v,\ v \in L^2)$ by applying the partial Fourier transformation and solving a certain simple ordinary differential equation. Christ's solutions are of the following form:

$$S_j^v(z, t) = \int_0^\infty e^{\imath t\tau} e^{-x^{2m}\tau} e^{\sigma(y)\imath a_j z \tau^{\frac{1}{2m}}} \tau^v d\tau,$$

for $y \neq 0$, $j \in \boldsymbol{N}$ and $v \geq 0$. Here $\pm ia_j$, $j \in \boldsymbol{N}$, are simple zeros of the function

$$\varphi(u) = \int_{-\infty}^\infty e^{-2(w^{2m} - uw)} dw.$$

It is known that the function φ has infinitely many zeros ([3]) and all of them exist on the imaginary axis ([8]). Thus we give them the order: $0 < a_j < a_{j+1}$ for $j \in \boldsymbol{N}$. It is easy to check that the S_j^v's are not real analytic on $\{(0 + iy, 0) ; y \in \boldsymbol{R}\}$. Besides this, the S_j^v's, off the set $\{y = 0\}$, belong to sth order Gevrey class G^s for all $s \geq 2m$, but no better, where $G^s := \{f ; \exists C > 0 \text{ s.t. } |\partial^\alpha f| \leq C^{|\alpha|}\Gamma(s|\alpha|)\ \forall \alpha\}$.

Let $S((z, t) ; (w, s))$ be the Szegö kernel of M; that is, the distribution kernel associated to the operator defined by the orthogonal projection