A Note on the Iwasawa λ-invariants of Real Quadratic Fields

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(Communicated by Shokichi IYANAGA, M. J. A., Jan. 12, 1996)

§1. Introduction For a number field k and a prime number p, denote respectively by $\lambda_p(k)$ and $\mu_p(k)$ the Iwasawa λ-invariant and the μ-invariant associated to the ideal class group of the cyclotomic $\boldsymbol{Z}_p$-extension over k. It is conjectured that $\lambda_p(k) = \mu_p(k) = 0$ for any totally real number field k and any p ([11, p. 316], [7]), which is often called Greenberg's conjecture. As for μ-invariants, we know that $\mu_p(k) = 0$ when k is an abelian field ([6]). The conjecture is still open even for real quadratic fields in spite of efforts of several authors (see Remark 2(2), Remark 3).

Let p be a fixed odd prime number and $k = \boldsymbol{Q}(\sqrt{d})$ a real quadratic field. Denote by χ the primitive Dirichlet character associated to k. Let $\lambda_p^*(k)$ be the λ-invariant of the power series associated to the p-adic L-function $L_p(s, \chi)$ (cf. [21, Thm. 7.10]). We have $\lambda_p(k) \leq \lambda_p^*(k)$ by the Iwasawa main conjecture (proved in [15]). So, $\lambda_p(k) = 0$ if $\lambda_p^*(k) = 0$. But, there are several examples with $\lambda_p^*(k) \geq 1$ (cf. [7, p. 266], [3]). Thus, it is natural to consider the following weak conjecture:

$$\lambda_p(k) \leq \max\{0, \lambda_p^*(k) - 1\}?$$

Let χ^* be the primitive Dirichlet character associated to $\omega\chi^{-1}$, where ω denotes the Teichmüller character $\boldsymbol{Z}/p\boldsymbol{Z} \to \boldsymbol{Z}_p$. When $\chi^*(p) = 1$, it is known that $\lambda_p^*(k) \geq 1$ and the weak conjecture is valid (see e.g. [10]).

The purpose of this note is to give some families of infinitely many real quadratic fields k with $\chi^*(p) \neq 1$ for which $\lambda_p^*(k) \geq 1$ and the weak conjecture is valid.

§2. Result/Remarks. Fix an odd prime number p and a square free natural number r with $\left(\frac{r}{p}\right) = -1$, where $\left(\frac{*}{p}\right)$ denotes the quadratic residue symbol. For each natural number m, we put

$$d_m^{(1)} = p^4 r^2 m^2 + r,\ d_m^{(2)} = p^4 m^2 + p.$$

Denote by $k_m^{(i)}$ the real quadratic field $\boldsymbol{Q}(\sqrt{d_m^{(i)}})$ $(i = 1,2)$. The prime p remains prime in $k_m^{(1)}$, and ramifies in $k_m^{(2)}$. Further, we have $\chi^*(p) \neq 1$ for these real quadratic fields. We prove the following

Proposition. *If $d_m^{(i)}$ is square free, then, $\lambda_p^*(k_m^{(i)}) \geq 1$ and the weak conjecture is valid for $k_m^{(i)}$ $(i = 1,2)$.*

Remark 1. Since the polynomial $p^4 r^2 X^2 + r$ (resp. $p^4 X^2 + p$) in X is irreducible in $\boldsymbol{Z}[X]$, there exist infinitely many m's for which $d_m^{(1)}$ (resp. $d_m^{(2)}$) is square free ([16], [17]).

Remark 2. (1) It is well-known that $\lambda_p(k) = 0$ for any quadratic field k such that $\left(\frac{k}{p}\right) \neq 1$ and $p \nmid h(k)$, $h(k)$ being the class number of k ([21, Thm. 10.4]). Let $p = 3$ and $r = 2$. Then, the family $\{k_m^{(1)}\}$ is "nontrivial" in the sense that we have several m satisfying the assumption of Proposition and $3 \mid h(k_m^{(1)})$, for example, $m = 1,3$. On the other hand, there are examples with $3 \nmid h(k_m^{(1)})$ such as $m = 2,4$. The family $\{k_m^{(1)}\}$ for $(p, r) = (5,2)$ and the family $\{k_m^{(2)}\}$ for $p = 3,5$ are also nontrivial. The author does not know, for $p \geq 7$, whether or not, the families given in Proposition are nontrivial. **(2)** It is proved that there exist infinitely many real quadratic fields k such that $\left(\frac{k}{3}\right) \neq 1$ and $3 \nmid h(k)$ ([18]). So, we have infinitely many real quadratic fields k with $\lambda_3(k) = 0$.

Remark 3. Several authors have given some criterions for the validity of Greenberg's conjecture or the weak conjecture (e.g. [4], [8], [9], [10], [12], [13], [14], [19], [20]). Using them, they have shown by some computation that $\lambda_3(k) = 0$ for many real quadratic fields k with "small" discriminants. The key lemma (Lemma 2) we use in the proof is one of the existing criterions.

§3. Proof of Proposition. Let k be a real quadratic field with a fundamental unit ε and χ the associated Dirichlet character. We need the following two lemmas.