Totally Real Minimal Submanifolds in a Quaternion Projective Space

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Abstract: In this paper, we prove some pinching theorems with respect to the scalar curvatures of 4-dimensional projectively flat (conharmonically flat) totally real minimal submanifolds in a 16-dimensional quaternion projective space.

Keywords: Totally real submanifold, Quaternion projective space, Curvature pinching

1. Introduction. A quaternion Kaehler manifold is defined as a $4n$-dimensional Riemannian manifold whose holonomy group is a subgroup of $Sp(1) \cdot Sp(n)$. A quaternion projective space $QP^n(c)$ is a quaternion Kaehler manifold with constant quaternion sectional curvature $c > 0$.

Let M be an n-dimensional Riemannian manifold and $J : M \to QP^n(c)$ an isometric immersion of M into $QP^n(c)$. If each tangent 2-subspace of M is mapped by J into a totally real plane of $QP^n(c)$, then M is called a totally real submanifold of $QP^n(c)$. Funabashi [3], Chen and Houh [1] and Shen [6] studied this submanifold and got some curvature pinching theorems. The purpose of this paper is to give some characterizations of 4-dimensional projectively flat (conharmonically flat) totally real minimal sub-manifolds in $QP^4(c)$.

2. Preliminaries. Let $QP^n(c)$ denote a $4n$-dimensional quaternion projective space with constant quaternion sectional curvature $c > 0$ and M be a totally real minimal submanifold in $QP^n(c)$ of dimension n. In this paper we will use the same notations and terminologies as in [1]. It was proved in [1] that the second fundamental form of the immersion satisfies

$$(2.1)\quad \frac{1}{2}\Delta \|\sigma\|^2 = \|\nabla'\sigma\|^2 + \sum tr(A_uA_v - A_vA_u)^2 - \sum (tr A_uA_v)^2 + \frac{c}{4}(n+1)\|\sigma\|^2$$

Since $\sum tr(A_uA_v - A_vA_u)^2$

$$= -\sum_{u,v,k,l}(\sum_m (h^u_{km}h^v_{lm} - h^v_{km}h^u_{lm}))^2$$

this together with the equation of Gauss, implies

$$(2.2)\quad \sum tr(A_uA_v - A_vA_u)^2 = -\|R\|^2 + c\rho - \frac{n-1}{8}nc^2.$$

Similarly, we have

$$(2.3)\quad \sum (tr A_uA_v)^2 = \|S\|^2 - \frac{n-1}{2}c\rho + n\left(\frac{n-1}{4}c\right)^2.$$

Combining (2.1) with (2.2), (2.3) and $\|\sigma\|^2 = \frac{c}{4}n(n-1) - \rho$, we obtain

$$(2.4)\quad \frac{1}{2}\Delta\|\sigma\|^2 = \|\nabla'\sigma\|^2 - \|R\|^2 - \|S\|^2 + \frac{n+1}{4}c\rho.$$

3. Projectively flat totally real minimal submanifold. Suppose M is an n-dimensional compact oriented totally real minimal submanifold in $QP^n(c)$, if M is projectively flat, then its projective curvature tensor $P^{[2]}$ satisfies

(3.1) $P(X, Y, Z, W) \overset{\text{def}}{=} R(X, Y, Z, W) - (g(X, W)S(Y, Z) - g(Y, W)S(X, Z))/(n-1) = 0$, where R, S, g are the curvature tensor, Ricci tensor and Riemannian metric of M respectively. From (3.1) we have

$$(3.2)\quad \|R\|^2 = \frac{2}{n-1}\|S\|^2$$

which, together with (2.4) asserts

$$(3.3)\quad \frac{1}{2}\Delta\|\sigma\|^2 = \|\nabla'\sigma\|^2 + \frac{n+1}{n-1}\|S\|^2 + \frac{n+1}{4}c\rho.$$

Taking the integrals of the both sides of (3.3) and using Green's theorem, we have

$$(3.4)\quad \int_M \|\nabla'\sigma\|^2 dV = \int_M \left(\|S\|^2/(n-1) - \frac{c}{4}\rho\right)(n+1)dV$$

On the other hand, by the Gauss-Bonnet theorem, when $n = 4$, the Euler number $\chi(M)$ of

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