Property C with Constraints for PDE

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Abstract: Let L_j, $j = 1,2$, be a pair of linear partial differential expressions in $\boldsymbol{R}^n$, $n \geq 3$, $D \subset \boldsymbol{R}^n$ be a bounded domain, $N_j := \{w : L_j w = 0 \text{ in } D\}$, N_{j,m_j} is a linear subspace in N_j of finite codimension $m_j < \infty$. We say that the pair $\{L_1, L_2\}$ has property C if the set of products $\{w_1 w_2\}$ is complete (total) in $L^p(D)$ for some $p \geq 1$. Here $w_j \in N_j$ run through subsets of N_j such that the products $w_1 w_2$ are well defined. We say that the pair $\{L_1, L_2\}$ has property C with constraints if the set $\{w_1 w_2\}$, where $w_j \in N_{j,m_j}$, $j = 1,2$, is total in $L^p(D)$. It is proved that if L_1 and L_2 have constant coefficients and the pair $\{L_1, L_2\}$ has property C then it has property C with constraints.

Key words: Property C with constraints; inverse problems; completeness of the set of products.

1. Introduction. The author introduced property C for pairs $\{L_1, L_2\}$ of linear partial differential expressions in [1] and has found many applications of this property [2]. In [3] he introduced property C with constraints and found several applications of this concept to inverse spectral problem, inverse boundary problem and inverse scattering problem.

In [2] necessary and sufficient conditions for property C to hold for a pair of linear partial differential expressions (formal differential operators) with constant coefficients are found.

The basic result of this paper is the following theorem.

Theorem 1.1. *If $\{L_1, L_2\}$ are linear formal partial differential operators in $\boldsymbol{R}^n$, $n \geq 3$, with constant coefficients and property C holds for the pair $\{L_1, L_2\}$, then property C with constraints holds for this pair.*

In section 2 we define property C and property C with constraints and recall some results from [2].

In section 3 we prove Theorem 1.1.

2. Basic definitions and known results.

2.1. Let $L_m u(x) = \sum_{|j| \leq J_m} a_{jm}(x) \partial^j u(x)$, $m = 1, 2, x \in \boldsymbol{R}^n$, $n \geq 2$, j is a multi-index, $a_{jm}(x)$ are given functions, $J_m > 0$ is an integer, $\partial^j u := \dfrac{\partial^j u}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}}$, $j_1 + \cdots + j_n := |j|$. We call L_m a linear formal differential operator.

Let $D \subset \boldsymbol{R}^n$ be a bounded domain, $N_m := \{w : L_m w = 0 \text{ in } D\}$, $f \in L^p(D)$, $p \geq 1$. The equation $L_m w = 0$ is understood in distributional sense. Assume that

$$\int_D f w_1 w_2 dx = 0, \quad w_m \in N_m \tag{2.1}$$

for all $w_m \in N_m$ for which $w_1 w_2 \in L^{p'}(D)$, $p' = \dfrac{p}{p-1}$.

Definition 2.1. *If* (2.1) *implies that* $f = 0$, *then we say that the pair* $\{L_1, L_2\}$ *has property* C.

Remark 2.1. *The name "property C" comes from "completeness of the set of products* $\{w_1 w_2\}$ [2].

We give now a necessary and sufficient condition for a pair $\{L_1, L_2\}$ of operators with constant coefficients, $a_{jm}(x) = a_{jm} = const$, to have property C.

Define

$$\mathscr{L}_m := \{z : z \in \boldsymbol{C}^n, L_m(z) = 0\}, \quad L_m(z) = \sum_{|j| \leq J_m} a_{jm} z^j. \tag{2.2}$$

Let $T_m(z_0)$ be the tangent space in $\boldsymbol{C}^n$ to the algebraic variety $\mathscr{L}_m$ at the point z_0.

Theorem 2.1. ([2,p.44]). *For a pair* $\{L_1, L_2\}$ *to have property* C *it is necessary and sufficient that there exist two points* $z_m \in L_m$, *such that the tangent spaces* $T_m(z_m)$, $m = 1,2$, *are transversal.*

Remark 2.2. *Geometrically this means that the variety* $\mathscr{L}_1 \cup \mathscr{L}_2$ *is not a union of parallel hyperplanes in* $\boldsymbol{C}^n$.

2.2. We now define *property C with con-*

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