On Some Examples of Modular QM-abelian Surfaces

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1. Introduction. Let $f=\sum_{n=1}^{\infty}a_nq^n$ be a (normalized) newform of weight two on $\Gamma_0(N)$ with trivial Nebentypus character such that the field of Fourier coefficients $K_f:=\boldsymbol{Q}(\{a_n\}_{n=1}^{\infty})$ is a (real) quadratic field. Let A_f denote the associated abelian surface over $\boldsymbol{Q}$ ([12], [13]). Then, $\mathrm{End}_{\boldsymbol{Q}}(A_f)\otimes\boldsymbol{Q}$, the $\boldsymbol{Q}$-algebra of endomorphisms of A_f over $\boldsymbol{Q}$, is exactly K_f. Let $\mathfrak{X}_f$ denote the $\boldsymbol{Q}$-algebra of *all* endomorphisms of A_f: $\mathfrak{X}_f=\mathrm{End}_{\bar{\boldsymbol{Q}}}(A_f)\otimes\boldsymbol{Q}$. If f is a form with complex multiplication, i.e., if there is a Dirichlet character $\psi\neq 1$ such that $a_p=\psi(p)a_p$ for all $p\nmid N$, then $A_f/\bar{\boldsymbol{Q}}$ is the product of two copies of an elliptic curve with complex multiplication by some imaginary quadratic field k, so that $\mathfrak{X}_f=\mathrm{M}_2(k)$. In the following, we always assume that f does not have complex multiplication (and that K_f is a real quadratic field). Then $\mathfrak{X}_f$ is either K_f, $\mathrm{M}_2(\boldsymbol{Q})$, or the quaternion division algebra B_D over $\boldsymbol{Q}$ with discriminant $D>1$ (see [7], [8]). We say that A_f has *quaternion multiplication* (or simply QM) if $\mathfrak{X}_f=B_D$ for some D.

Definition. Let $f=\sum a_nq^n$ be as above and let χ be a (primitive) Dirichlet character. Then f is said to possess the *extra twist* by χ if the equality

$$a_p^{\sigma}=\chi(p)a_p$$

holds for all $p\nmid N$, where σ is the non-trivial automorphism of $K_f/\boldsymbol{Q}$. In this case, we say that χ is a *twisting character* of f.

Let f be a newform on $\Gamma_0(N)$ satisfying our assumption. Then $f^{\sigma}:=\sum a_n^{\sigma}q^n$ is also a newform on $\Gamma_0(N)$. Further, if χ is any primitive quadratic Dirichlet character of conductor r, then $f^{\chi}:=\sum a_n\chi(n)q^n$ is a cuspform on $\Gamma_0(N')$, where N' is the least common multiple of N and r^2. See [13] for general background.

Now let f be a newform on $\Gamma_0(N)$ which possesses the extra twist by χ, say. Then χ is quadratic and the square of the conductor of χ divides N, and in fact $f^{\sigma}=f^{\chi}$. It is also easily seen that χ is a unique twisting character of f, since f is a form without complex multiplication.

Proposition 1. *Let f possess the extra twist by χ. Then*

$$\mathfrak{X}_f=\left(\frac{d,\ \chi(-1)r}{\boldsymbol{Q}}\right),$$

where $\left(\frac{a,\ b}{\boldsymbol{Q}}\right)$ is the quaternion algebra over $\boldsymbol{Q}$ with reduced norm form $x^2-ay^2-bz^2+abw^2$, d is the discriminant of K_f and r is the conductor of χ.

Proof. This is a special case of a result of [7], [8]. □

If f does not possess the extra twist, it is known that $\mathfrak{X}_f=K_f$.

Proposition 2. *Let A_f be an abelian surface attached to a newform f of weight two on $\Gamma_0(N)$ and assume that A_f has QM. Let p be a prime divisor of N with $p^{\nu}\parallel N$. Then*

(1) $2\leq\nu\leq 10$ *if* $p=2$,

(2) $2\leq\nu\leq 5$ *if* $p=3$,

(3) $\nu=2$ *if* $p\geq 5$.

Furthermore, N is divisible by 2^5 or by the square of some prime p such that $p\equiv 3 \pmod 4$.

Proof. By assumption, f possesses the extra twist. If N is exactly divisible by a prime, then $\mathfrak{X}_f=\mathrm{M}_2(\boldsymbol{Q})$ by [9], Theorem 2. So $\nu\geq 2$ if A_f has QM. Put

$$s=\left\lceil\frac{\nu}{2}-1-\frac{1}{p-1}\right\rceil,$$

where $\lceil x\rceil$ is the least integer $\geq x$. Then by [3], Theorem 5.5, the center of $\mathfrak{X}_f$ contains $\boldsymbol{Q}(\zeta+\zeta^{-1})$ if $p>2$ (resp. $\boldsymbol{Q}(\zeta^2+\zeta^{-2})$ if $p=2$), where ζ is a primitive p^s-th root of unity, hence we obtain the estimate for ν. The last part follows from [9], Theorem 3 and [1], Theorem 7. □

An example of a QM-abelian surface attached to a newform of weight two on $\Gamma_0(N)$ is given by Koike [6]. In this case the level is $243=3^5$, $K_f=\boldsymbol{Q}(\sqrt{6})$, $\chi=\left(\frac{-3}{\cdot}\right)$ and $\mathfrak{X}_f=\left(\frac{6,\ -3}{\boldsymbol{Q}}\right)=B_6$. Since there are, as it seems, no other known examples, it might be interesting to find other examples of modular QM-abelian sur-