Hecke Correspondences and Betti Cohomology Groups for Smooth Compactifications of Hilbert Modular Varieties of Dimension ≤ 3

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Abstract: We consider Betti cohomology groups of smooth toroidal compactifications of Hilbert modular varieties and representations of Hecke correspondences on them. We study absolute values and zeta functions of eigenvalues of those operators for varieties of dimension ≤ 3. Our main results are Theorems $1 \sim 6$ below.

Notations and introduction. K : a totally real algebraic number field, $\mathcal{O}$ = the principal order of K, $g = [K : \boldsymbol{Q}]$, N : a rational integer ≥ 3, $\Gamma(1) = \{\gamma \in \mathrm{GL}_2(\mathcal{O}) \mid \det\gamma$ is totally positive$\}$, $\Gamma(N) = \{\gamma \in \Gamma(1) \mid \gamma \equiv 1 \ (\mathrm{mod}\, N\mathcal{O})\}$, $\mathfrak{H}^g$: the Cartesian product of g complex upper half planes, $\Gamma(N)\backslash\mathfrak{H}^g$: the Hilbert modular variety. We fix a regular and projective $\Gamma(1)$-admissible family Σ of polyhedral cone decompositions once for all. Note "$\Gamma(1)$-admissible" = "((the totally positive units group of $\mathcal{O}$) $\ltimes$ (the additive group $\mathcal{O}$))-admissible". For a neat congruence subgroup Γ of $\Gamma(1)$, from Σ, one gets the smooth projective toroidal compactification $M_\Gamma = (\Gamma\backslash\mathfrak{H}^g)^{\sim}$ of $\Gamma\backslash\mathfrak{H}^g$, cf. Ash et al. [1], Hirzebruch [10].

This note may be regarded as continuation of [8] and [9]. In Theorem 2 of [8] and Theorem 8 of [9] we have given sharp estimates for eigenvalues of "Hecke operators" acting on Betti cohomology groups of arbitrary degrees $d \geq 0$ of smoothly compactified Hilbert modular varieties for all the prime ideals $p\mathcal{O}$ of $\mathcal{O}$ with prime numbers $p \nmid N$, also cf. Remark 2 below. We shall study them on middle Betti cohomology groups also for the other prime ideals of $\mathcal{O}$ in Theorems 1 and 5 ($g \leq 3$). For this we shall extend the method given in [7], cf. Theorems $1 \sim 3$ in §1. In addition, for any $g > 0$ we shall study "Hecke operators" $\{F_n(T(\mathfrak{U}))\}_{(\mathfrak{U},N)=1}$ with $n \geq 0$ defined adelicly and acting on certain direct sum of Betti cohomology groups of the smooth compactifications, cf. Theorem 4 and the explanation before Theorem 6. We shall consider also zeta functions with Euler products attached to arbitrary common eigen-forms for $\{F_n(T(\mathfrak{U}))\}_{(\mathfrak{U},N)=1}$, cf. Theorem 6.

§1. Treatment without adeles. Write $G^+(\mathcal{O})$ = the monoid $\{\gamma \in M_{2,2}(\mathcal{O}) \mid \det\gamma$ is totally positive$\}$, and D = the Hecke ring $\mathrm{HR}(\Gamma(N), G^+(\mathcal{O}))$. Write C = the algebraic correspondence ring of the cycles of codimension g on $M_{\Gamma(N)} \times_{\mathrm{Spec}\, \boldsymbol{C}} M_{\Gamma(N)}$. Let $\alpha \in G^+(\mathcal{O})$. Put $a = N_{\boldsymbol{Q}}^K(\det \alpha)$. Recall [8] and [9]. The complex analytic morphism $\alpha^\vee : \Gamma(a^2N)\backslash\mathfrak{H}^g \to \Gamma(N)\backslash\mathfrak{H}^g$ induced by $\alpha : \mathfrak{H}^g \to \mathfrak{H}^g$ and $\alpha\Gamma(a^2N)\alpha^{-1} \subset \Gamma(N)$, extends to a unique morphism $\varphi\circ\alpha^{\sim}\circ\varphi_1 : M_{\Gamma(a^2N)} \to M_{\Gamma(N)}$ (see the explanation next to (2) below.) Let can denote the canonical morphism: $M_{\Gamma(a^2N)} \to M_{\Gamma(N)}$ induced by id.: $\mathfrak{H}^g \to \mathfrak{H}^g$ and $\Gamma(a^2N) \subset \Gamma(N)$. Let $\mathfrak{z}(\Gamma(N)\alpha\Gamma(N))$ denote the scheme theoretic image of the morphism (can, $\varphi\circ\alpha^{\sim}\circ\varphi_1$): $M_{\Gamma(a^2N)} \to M_{\Gamma(N)} \times_{\mathrm{Spec}\, \boldsymbol{C}} M_{\Gamma(N)} = M_{\Gamma(N)}^2$. It is a cycle of codimension g. By [4], [6] and [9], the map $F : D \to C$, given by $\Gamma(N)\alpha\Gamma(N) \mapsto \mathfrak{z}(\Gamma(N)\alpha\Gamma(N))$, is a ring homomorphism. Let $\xi : C \to H^{2g}(M_{\Gamma(N)}^2, \boldsymbol{C})$ denote the ring homomorphism given by $y \mapsto$ the fundamental class of y, cf. [9], §5. By Künneth we have the anti-$\boldsymbol{C}$-algebra isomorphism: $H^{2g}(M_{\Gamma(N)}^2, \boldsymbol{C}) \cong \Pi_{n=0}^{2g} \mathrm{End}_{\boldsymbol{C}} H^n(M_{\Gamma(N)}, \boldsymbol{C})$, $Z \mapsto (\rho_n(Z))_{n=0}^{2g}$, cf. [9], §5. Put $\Psi = \rho_g\circ\xi\circ F$. We get the anti-$\boldsymbol{C}$-algebra homomorphism $\psi = (\mathrm{Id.}) \otimes \Psi : \boldsymbol{C} \otimes_{\boldsymbol{Z}} D \to \mathrm{End}_{\boldsymbol{C}} H^g(M_{\Gamma(N)}, \boldsymbol{C})$ by the scalar extension of Ψ from $\boldsymbol{Z}$ to $\boldsymbol{C}$. For any $x \in \boldsymbol{C} \otimes_{\boldsymbol{Z}} D$, we call $\psi(x)$ the Hecke operator of x on $H^g(M_{\Gamma(N)}, \boldsymbol{C})$, cf. [5], [6], [8] and [9]. Our first main result is

Theorem 1. *Assume* $g \leq 3$. *Write* $\Gamma = \Gamma(N)$. *Let r be an integer* > 0, *let* $\{w_i\}_{i=1}^r$ *be complex numbers* $\neq 0$, *and let* $\{\mathcal{A}_i\}_{i=1}^r \subset G^+(\mathcal{O})$. *Let λ denote any eigenvalue of* $\psi(\sum_{i=1}^r w_i \cdot \Gamma\mathcal{A}_i\Gamma)$. *We*