Construction of Jacobi Forms from Certain Combinatorial Polynomials

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Introduction. The concept of Jacobi polynomials as a certain generalization of weight enumerators of certain codes was introduced by Ozeki [9]. In this paper we first interprete these Jacobi polynomials (defined by Ozeki) as the homogeneous polynomials of 4 variables which are invariant under the 4-dimensional action of the 2-dimentional finite unitary reflection group of order 192 (No. 9 in Shephard-Todd [12]). Then we determine the Molien series and the structure of the invariant ring. In Section 2, we define a new map from the space of homogeneous Jacobi polynomials (i.e., the invariant ring) to a space of Jacobi forms (in the sense of Eichler-Zagier [5]). This map, which we believe is very important for the study of Jacobi forms, is an extension of the well-known Broué-Enguehard map (and the extended version of it due to Ozeki [9], see also [4, Proposition 5.4]) from the space of the weight enumerators of binary self-dual doubly even codes to a space of modular forms. (Note that modular forms are Jacobi forms of index 0.) We conclude this paper by mentioning the outlines of further generalizations concerning this new map. The purpose of this paper is to announce the new results. The details will be published in forthcoming papers which are in preparation.

We remark that the papers [7], [3], [11] also deal with extensions of Broué-Enguehard theorem in various directions, in particular to obtain Siegel modular forms.

1. The space of Jacobi polynomials. Let $G = \langle \sigma_1, \sigma_2 \rangle$ with

$$(1) \qquad \sigma_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & & \\ 1 & -1 & & \\ & & 1 & 1 \\ & & 1 & -1 \end{bmatrix}$$

$$\text{and } \sigma_2 = \begin{bmatrix} \sqrt{-1} & & & \\ & 1 & & \\ & & \sqrt{-1} & \\ & & & 1 \end{bmatrix}$$

be the subgroup of $GL(4, \mathrm{C})$ generated by the two elements σ_1 and σ_2. Then $|G| = 192$ and G is the finite unitary group generated by reflections (u.g.g.r) referred to as No. 9 in Shephard-Todd [12]. Let $R = \mathrm{C}[x_1, y_1, x_2, y_2]$ and let R^G be the ring of invariants under the group $G = \langle \sigma_1, \sigma_2 \rangle$ with the natural action (1). Then we can interprete R^G as the space of Jacobi polynomials (in the sense of Ozeki [9]) for binary self-dual doubly even codes.

To be more precise, let V be a vector space of dimension n over the binary field $GF(2)$. V is equipped with the usual inner product $u \cdot v = u_1v_1 + u_2v_2 + \cdots + u_nv_n$ in $GF(2)$ for $u = (u_1, v_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$. We also define

$$u * v = \#\{j \mid 1 \leq j \leq n, u_j = v_j = 1\}.$$

Note that $u * u$ is the Hamming weight $w(u)$ of u in V. Let $\mathscr{C}$ be a binary self-dual doubly even code in V, i.e., $\mathscr{C}$ is a vector subspace (over $GF(2)$) of V satisfying

$$\mathscr{C} = \mathscr{C}^{\perp} := \{y \in V \mid x \cdot y = 0, \forall x \in \mathscr{C}\}$$

and $4 \mid w(u)$ for $\forall u \in \mathscr{C}$. For a binary self-dual doubly even code $\mathscr{C}$ in V and for a vector v in V, Ozeki [9] defines the polynomial $J(\mathscr{C}, v \mid X, Z)$ in X and Z by

$$J(\mathscr{C}, v \mid X, Z) = \sum_{u \in \mathscr{C}} X^{u*u} Z^{u*v}.$$

For each of such polynomial $J(\mathscr{C}, v \mid X, Z)$, we can naturally associate a homogeneous polynomial $J(\mathscr{C}, v \mid x_1, y_1, x_2, y_2)$ of degree n in x_1, y_1, x_2, y_2 by

$$J(\mathscr{C}, v \mid x_1, y_1, x_2, y_2) = \sum_{u \in \mathscr{C}} x_1^{u*u-u*v} y_1^{n-v*v-(u*u-u*v)} x_2^{u*v} y_2^{v*v-u*v}.$$

Note that the correspondence between the homogeneous and inhomogeneous Jacobi polynomials is one to one, and this correspondence gives an analogy with the one between the

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