Orbits in the Flag Variety and Images of the Moment Map for $U(p, q)$

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(Communicated by Shokichi IYANAGA, M. J. A., June 11, 1996)

1. Introduction. Let $G_{\boldsymbol{R}}$ be a real classical group. For simplicity, we consider only the case that Cartan subgroups of $G_{\boldsymbol{R}}$ are connected. It is known that there is a correspondence between irreducible representation of $G_{\boldsymbol{R}}$ with a given infinitesimal character and the orbits of a flag of the complexification of $G_{\boldsymbol{R}}$ on which the complexification K of a maximal compact subgroup of $G_{\boldsymbol{R}}$ operates (for example [7]). Essentially, a parametrization of the orbits is given by Matsuki-Oshima [5]. In this paper, we will consider the case $G_{\boldsymbol{R}} = U(p, q)$ and get an algorithm which gives representatives of orbits for the parametrization and images of the moment map of the conormal bundles of orbits. For a closed orbit, the image is the associated variety of a representation [1]. By the method in this paper, we get representatives from parameters directly. The proof will appear in another paper. Garfinkle gave another algorithm in [3]. By her algorithm we can also get signed Young diagram from a parameter of K-orbits in the flag variety. It remains to examine how two algorithms agree.

The argument in this paper can be applied to the case $G_{\boldsymbol{R}} = Sp(p, q)$, etc. The concerning results will appear elsewhere.

Notation 1.1. Let $\boldsymbol{N}$ denote the set of positive integers; $\boldsymbol{N} = \{1,2,\ldots\}$. For $n \in \boldsymbol{N}$, let an $n \times n$ matrix $E_{ij} (1 \leq i, j \leq n)$ denote the matrix unit which has 1 for the (i, j)-entry and 0 for other entries. Let an n-column vector e_i be the vector which has 1 for the i-th entry and 0 for other entries. For a matrix A, let A_{st} be the (s, t)-entry of A. Let $\mathrm{Mat}(m, n)$ be the set of $m \times n$-matrices over $\boldsymbol{C}$. Let $I_n \in \mathrm{Mat}(n, n)$ be the identity matrix. For an $A \in \mathrm{Mat}(n, n)$ and a subset $\{i(1), i(2), \ldots, i(m)\}$ of $\{1, \ldots, n\}$, we denote by $A_{(i(1), i(2), \ldots, i(m))}$ an $m \times m$-matrix whose (s, t)-entry is $A_{i(s)i(t)}$. Let $\#S$ denote cardinality of the finite set S. For m vectors $\{g_1, \ldots, g_m \mid g_i \in \boldsymbol{C}^n\}$ $(m < n)$ let $\langle g_1, \ldots, g_m \rangle$ be vector space spanned by $\{g_1, \ldots, g_m\}$. Let $\mathfrak{S}_n$ be the set of permutations of $\{1, \ldots, n\}$.

2. A symbolic parametrization of K-orbits. Let $G_{\boldsymbol{R}}$ be a real classical Lie group with Lie algebra $\mathfrak{g}_{\boldsymbol{R}}$, G the complexification of $G_{\boldsymbol{R}}$, θ a Cartan involution of $\mathfrak{g}_{\boldsymbol{R}}$. Let $\mathfrak{g}_{\boldsymbol{R}} = \mathfrak{k}_{\boldsymbol{R}} + \mathfrak{p}_{\boldsymbol{R}}$ be the Cartan decomposition corresponding to θ, $\mathfrak{k}$ the complexification of $\mathfrak{k}_{\boldsymbol{R}}$, K the analytic subgroup of G for $\mathfrak{k}$, and B a Borel subgroup of G. Although we restrict ourselves to the case $G_{\boldsymbol{R}} = U(p, q)$ in the main body of the paper, a preliminary discussion holds for an arbitrary linear connected reductive Lie group $G_{\boldsymbol{R}}$.

In this section we recall a symbolic parametrization of K-orbits in X.

Let $n = p + q$. We realize the indefinite unitary group $G_{\boldsymbol{R}} = U(p, q)$ as the group of matrices g in $GL(n, \boldsymbol{C})$ which leave invariant the Hermitian from of the signature (p, q)

$$x_1\overline{x_1} + \cdots + x_p\overline{x_p} - x_{p+1}\overline{x_{p+1}} - \cdots - x_n\overline{x_n},$$

i.e.,

$$U(p, q) = \left\{ g \in GL(n, \boldsymbol{C}) \mid {}^t g \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \bar{g} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \right\}.$$

We fix a Cartan involution θ of $G_{\boldsymbol{R}}$:

$$\theta : x \mapsto \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} x \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}.$$

Definition 2.1 (Clan) (see [5]). An *indication* for $U(p, q)$ is an ordered set $(c_1 \ldots c_n)$ of n symbols satisfying the following four conditions.

1. For every $1 \leq i \leq n$, c_i is $+$, $-$, or an element of $\boldsymbol{N}$.
2. If $c_i \in \boldsymbol{N}$, then there exists a unique $j \neq i$ with $c_j = c_i$, i.e.,
   $\#\{i \mid c_i = a\} = 0$ or 2 for any $a \in \boldsymbol{N}$.
3. The difference between numbers of $+$ and $-$ in an indication $(c_1 \ldots c_n)$ coincides with the difference of signatures of the Hermitian form defining the group $G_{\boldsymbol{R}}$: